THE COMPLETENESS THEOREM FOR MONADS IN CATEGORIES OF SORTED SETS

J. CLIMENT VIDAL AND J. SOLIVERES TUR

Abstract. The completeness theorem of equational logic of Birkhoff asserts the coincidence of the model-theoretic and proof-theoretic consequence relations. Goguen and Meseguer, giving a sound and adequate system of inference rules for many-sorted deduction, generalized the completeness theorem of Birkhoff to the completeness of many-sorted equational logic and provided simultaneously a full algebraization of many-sorted equational deduction. In this paper, after simplifying the presentation of Hall algebras and the inference rules given by Goguen-Meseguer, we give another proof of the completeness theorem by using the Bénabou algebras and, once defined the concepts of equational class and equational theory for a monad in a category and the concept of $\varprojlim$-compatible congruence on a category, we prove that the lattice of Π-compatible congruences on the category of polynomials for a monad in a category of sorted sets is identical to the lattice of equational theories for the same monad. In this way we obtain a completeness theorem for monads in categories of sorted sets, hence independent of any explicit syntactical representation of the relevant concepts, that generalizes the completeness theorem of Goguen-Meseguer and provides a full categorization of many-sorted equational deduction.

1. Introduction.

The completeness theorem of many-sorted equational logic of Goguen-Meseguer, see [5], under which falls the classical completeness theorem of equational logic of Birkhoff, see [2], asserts the coincidence of two closure operators on the set $\mathrm{Eq_H}(\Sigma)$ of finitary Σ-equations, for an S-sorted signature Σ and an S-sorted set of variables $V = (V_s)_{s\in S}$ where, for every s in S, $V_s = \{\, v^s_n \mid n \in \mathbb{N}\}$. One of the closure operators, the semantical consequence operator, denoted by Cn_Σ, is obtained from the contravariant Galois connection between the ordered class $\underline{\mathrm{Sub}}(\mathbf{Alg}(\Sigma))$, of subclasses of $\mathbf{Alg}(\Sigma)$, and the ordered set $\underline{\mathrm{Sub}}(\mathrm{Eq_H}(\Sigma))$, of subsets of $\mathrm{Eq_H}(\Sigma)$, composing the operators $\mathrm{Mod}_\Sigma\colon \underline{\mathrm{Sub}}(\mathrm{Eq_H}(\Sigma)) \longrightarrow \underline{\mathrm{Sub}}(\mathbf{Alg}(\Sigma))$ and $\mathrm{Th}_\Sigma\colon \underline{\mathrm{Sub}}(\mathbf{Alg}(\Sigma)) \longrightarrow \underline{\mathrm{Sub}}(\mathrm{Eq_H}(\Sigma))$, obtained from the ternary satisfiability relation between valuations, many-sorted Σ-algebras and finitary Σ-equations. The other closure operator, the formal consequence operator, can be obtained not only from axioms and inference rules but also, alternatively, as has been pointed out by Goguen and Meseguer in [5], as the operator $\mathrm{Cg}_{\underline{\mathrm{Pol}}_\mathrm{H}(\Sigma)}$, of generated congruence, on the Hall algebra $\underline{\mathrm{Pol}}_\mathrm{H}(\Sigma)$ that has as underlying $S^\star \times S$-sorted set $(\mathrm{Fr}_\Sigma(\downarrow w)_s)_{(w,s)\in S^\star\times S}$ where, for $w \in S^\star$, $\downarrow w$ is the S-sorted set that has as s-th coordinate the subset $\{\, v^s_i \in V_s \mid w(i) = s\,\}$ of V_s while $\mathrm{Fr}_\Sigma(\downarrow w)$ is the underlying S-sorted set of $\underline{\mathrm{Fr}}_\Sigma(\downarrow w)$, the free many-sorted Σ-algebra on $\downarrow w$. For this alternative

Date: May 26, 2005.

1991 *Mathematics Subject Classification.* Primary: 03C05, 08A68, 08B05, 18A32, 18C15, 18C20; Secondary: 06B23.

Key words and phrases. Hall algebra, Bénabou algebras, many-sorted algebra, polynomial for a monad, equation for a monad, $\varprojlim$-compatible congruence, Π-compatible congruence, completeness for monads.

point of view it is essential to conceive of the sets of finitary Σ-equations as subsets of the square of the Hall algebra $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$, i.e., to think of such sets as parts of the square of an algebraic construct and not of an unstructured object as in the first alternative. This point of view allows a *full algebraization of many-sorted equational deduction*.

In the second section, after simplifying the presentation of Hall algebras and the inference rules given by Goguen-Meseguer, we define the concept of Bénabou algebra, in order to give another proof of the completeness theorem, and prove that the category of Bénabou theories, defined in [1], is isomorphic to the category of Bénabou algebras and also that the category of Hall algebras, used by Goguen-Meseguer in their proof, is equivalent to that of Bénabou algebras. We point out that in [3] the Bénabou algebras have also been used to define what we have called morphisms of Fujiwara from a many-sorted signature into another (such morphisms consists of two suitably related mappings: On the one hand, a mapping that relates the sets of sorts of the many-sorted signatures and assigns to each sort in the first, a derived sort in the second, i.e., a word on the set of sorts of the second, and, on the other hand, a mapping that assigns to each operation in the first, a family of many-sorted polynomials in the second, all in such a way that both transformations are compatible), as well as morphisms from a many-sorted specification into another, and we remark that the hypersubstitutions are a particular case of the above morphisms between many-sorted signatures.

Now, if we consider that a monad (exactly, those that arise from an algebraic adjunction) is what remains invariant under change of algebraic presentation or, in other words, if we take into account the equivalence between monads and theories, then it seems natural to intend to prove directly a completeness theorem for monads, hence independent of any explicit syntactical representation of the relevant concepts. But, as we will see, it happens that such a direct proof is not an automatic translation of the above mentioned proofs.

In the third section, in order to obtain a direct completeness theorem for monads, not necessarily finitary, we define polynomials, equations and validity for monads. Once this is done we also obtain a contravariant Galois connection between the ordered class $\underline{\mathrm{Sub}}(\mathbf{EM}(\mathbb{T}))$, of subclasses of $\mathbf{EM}(\mathbb{T})$, the Eilenberg-Moore category for the monad $\mathbb{T}$, and the ordered class $\underline{\mathrm{Sub}}(\mathrm{Eq}(\mathbb{T}))$, of subclasses of $\mathrm{Eq}(\mathbb{T})$, the equations for the monad $\mathbb{T}$, from such a connection we obtain the semantical consequence operator, $\mathrm{Cn}_{\mathbb{T}}$, on $\mathrm{Eq}(\mathbb{T})$ composing $\mathrm{Mod}_{\mathbb{T}}\colon \underline{\mathrm{Sub}}(\mathrm{Eq}(\mathbb{T})) \longrightarrow \underline{\mathrm{Sub}}(\mathbf{EM}(\mathbb{T}))$ and $\mathrm{Th}_{\mathbb{T}}\colon \underline{\mathrm{Sub}}(\mathbf{EM}(\mathbb{T})) \longrightarrow \underline{\mathrm{Sub}}(\mathrm{Eq}(\mathbb{T}))$.

In the last section, in order to obtain the missing formal consequence operator on $\mathrm{Eq}(\mathbb{T})$ we define the concept of Π-compatible congruence on a category and take into account that $\mathrm{Eq}(\mathbb{T})$ is a subfamily of the square of the family of the hom-sets of the category $\mathbf{Pol}(\mathbb{T})$ of polynomials for $\mathbb{T}$, the dual of the Kleisli category $\mathbf{Kl}(\mathbb{T})$ of the monad $\mathbb{T}$, and from this the formal consequence operator arises as the operator $\mathrm{Cg}^{\Pi}_{\mathbf{Pol}(\mathbb{T})}$, generated Π-compatible congruence, on $\mathbf{Pol}(\mathbb{T})$. Finally, the completeness theorem for monads in categories of sorted sets asserts the coincidence between both operators or, what amounts to the same, that the lattice of Π-compatible congruences on the category of polynomials for a monad in a category of sorted sets is identical to the lattice of equational theories for the monad.

In this way the completeness theorem of many-sorted equational logic of Goguen-Meseguer and the classical completeness theorem of equational logic of Birkhoff, are instances of this completeness theorem and this last is, in addition, invariant under presentations. We believe that from the above we obtain a *full categorization of many-sorted equational deduction*.

In what follows we use standard concepts from many-sorted algebra and category theory, see e.g., [5] for many-sorted algebra and [6] for category theory. Moreover, every set we consider will be an element or subset of a Grothendieck universe $\boldsymbol{\mathcal{U}}$, fixed once and for all.

2. Hall algebras, the many-sorted completeness of Goguen-Meseguer and Bénabou algebras.

The Hall algebras formalize the concept of substitution for the finitary polynomials and that of generalized composition for the many-sorted operations on a sorted set. In this section we define the variety of Hall algebras, through an axiom system less redundant than that presented in [5] and, from the completeness theorem of many-sorted equational logic, we obtain a many-sorted equational calculus from which we prove that the rules of abstraction and concretion in [5] are derived rules, hence providing a somewhat less redundant set of sound and adequate inference rules than in [5]. Moreover, once defined the category of Bénabou algebras we prove that it is isomorphic to the category of Bénabou theories in [1], that the category of Hall algebras is equivalent to the category of Bénabou algebras and, additionally, we give an alternative proof of the Completeness Theorem through the Bénabou algebras.

But before that we consider, for a set of sorts S and an S-sorted signature Σ, the concepts of finitary Σ-polynomial, finitary Σ-equation and the relation of validation between finitary Σ-equations and Σ-algebras. From these concepts we obtain, as is well known, a contravariant Galois connection between the ordered set of families of finitary Σ-equations and the ordered class of families of Σ-algebras and, in particular, the closure operator of semantical consequence on the set of finitary Σ-equations.

Definition 1. Let Σ be an S-sorted signature, $w \in S^{\star}$ and $s \in S$.

(1) A *finitary Σ-polynomial of type* (w, s) is a mapping $P\colon \delta^s \longrightarrow \mathrm{Fr}_\Sigma(\downarrow w)$ where $\delta^s = (\delta^s_t)_{t\in S}$, the delta of Kronecker in s, is such that $\delta^s_t = \varnothing$ if $s \neq t$ and $\delta^s_s = 1$.

(2) A *finitary Σ-equation of type* (w, s) is a pair (P, Q) of finitary Σ-polynomials of type (w, s).

We agree that $\mathrm{Pol}_{\mathrm{H}}(\Sigma)$ denotes the many-sorted set of finitary Σ-polynomials, i.e., $(\mathrm{Hom}_{\mathbf{Set}^S}(\delta^s, \mathrm{Fr}_\Sigma(\downarrow w)))_{(w,s)\in S^\star\times S}$, identified to $(\mathrm{Fr}_\Sigma(\downarrow w)_s)_{(w,s)\in S^\star\times S}$. On the other hand, $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ denotes the many-sorted set of finitary Σ-equations, i.e., $(\mathrm{Hom}_{\mathbf{Set}^S}(\delta^s, \mathrm{Fr}_\Sigma(\downarrow w))^2)_{(w,s)\in S^\star\times S}$, identified to $(\mathrm{Fr}_\Sigma(\downarrow w)^2_s)_{(w,s)\in S^\star\times S}$. We point out that the above identifications can be made because the deltas of Kronecker are a system of generators for the category $\mathbf{Set}^S$.

Now we define for an S-sorted signature Σ, on the one hand, the realization of the finitary Σ-polynomials in the Σ-algebras and, on the other, the concept of validation of a finitary Σ-equation in a Σ-algebra.

Definition 2. Let Σ be an S-sorted signature, $w \in S^\star$, $s \in S$ and $\underline{A}$ a Σ-algebra. Then every finitary Σ-polynomial $P\colon \delta^s \longrightarrow \mathrm{Fr}_\Sigma(\downarrow w)$ determines a mapping $P^{\underline{A}}$ from $\mathrm{Hom}_{\mathbf{Set}^S}(\downarrow w, A)$ to $\mathrm{Hom}_{\mathbf{Set}^S}(\delta^s, A)$, the *realization* of P in $\underline{A}$, that to a mapping $f\colon \downarrow w \longrightarrow A$ assigns $f^\sharp \circ P\colon \delta^s \longrightarrow A$ where $f^\sharp$ is the canonical extension of f to $\underline{\mathrm{Fr}}_\Sigma(\downarrow w)$.

Definition 3. Let $\underline{A}$ be a Σ-algebra and (P, Q) a finitary Σ-equation of type (w, s). We say that (P, Q) is *valid* in $\underline{A}$, denoted by $\underline{A} \models^\Sigma_{w,s} (P, Q)$, if $P^{\underline{A}} = Q^{\underline{A}}$. If $\mathcal{K} \subseteq \mathbf{Alg}(\Sigma)$, then we agree that $\mathcal{K} \models^\Sigma_{w,s} (P, Q)$ means that, for every $\underline{A} \in \mathcal{K}$, $\underline{A} \models^\Sigma_{w,s} (P, Q)$.

We remark that the underlying reason for the definition we have made of the finitary Σ-polynomials, the realization of finitary Σ-polynomials and the validation relation, will become clear when these concepts be compared with the corresponding ones, in a later section, for a monad $\mathbb{T}$ in a category.

From the concept of validation we obtain the following contravariant Galois connection.

Definition 4. Let Σ be an S-sorted signature.

(1) If $\mathcal{K} \subseteq \mathbf{Alg}(\Sigma)$, then $\mathrm{Th}_\Sigma(\mathcal{K})$, the *finitary Σ-equational theory determined by $\mathcal{K}$*, has as elements the finitary Σ-equations $(P,Q)\colon \delta^s \longrightarrow \mathrm{Fr}_\Sigma({\downarrow}w)$ such that $\mathcal{K} \models^\Sigma_{w,s} (P,Q)$, i.e.,

$$\mathrm{Th}_\Sigma(\mathcal{K}) = \big(\{(P,Q) \in \mathrm{Eq}_\mathrm{H}(\Sigma)_{w,s} \mid \forall \underline{A} \in \mathcal{K}\ (\underline{A} \models^\Sigma_{w,s} (P,Q))\}\big)_{(w,s)\in S^\star\times S}$$

(2) If $\mathcal{E} \subseteq \mathrm{Eq}_\mathrm{H}(\Sigma)$, then $\mathrm{Mod}_\Sigma(\mathcal{E})$, the *finitary Σ-equational class determined by $\mathcal{E}$*, has as elements the Σ-algebras $\underline{A}$ that validate each equation of $\mathcal{E}$, i.e.,

$$\mathrm{Mod}_\Sigma(\mathcal{E}) = \left\{ \underline{A} \in \mathbf{Alg}(\Sigma) \;\middle|\; \begin{array}{c} \forall (w,s) \in S^\star \times S,\ \forall (P,Q) \in \mathcal{E}_{w,s}, \\ \underline{A} \models^\Sigma_{w,s} (P,Q) \end{array} \right\}$$

Proposition 1. *Let Σ be an S-sorted signature, $\mathcal{E}$, $\mathcal{E}'$ two families of finitary Σ-equations and $\mathcal{K}$, $\mathcal{K}'$ two classes of Σ-algebras. Then the following holds:*

(1) *If $\mathcal{E} \subseteq \mathcal{E}'$, then $\mathrm{Mod}_\Sigma(\mathcal{E}') \subseteq \mathrm{Mod}_\Sigma(\mathcal{E})$.*
(2) *If $\mathcal{K} \subseteq \mathcal{K}'$, then $\mathrm{Th}_\Sigma(\mathcal{K}') \subseteq \mathrm{Th}_\Sigma(\mathcal{K})$.*
(3) *$\mathcal{E} \subseteq \mathrm{Th}_\Sigma(\mathrm{Mod}_\Sigma(\mathcal{E}))$ and $\mathcal{K} \subseteq \mathrm{Mod}_\Sigma(\mathrm{Th}_\Sigma(\mathcal{K}))$.*

Therefore the pair of mappings Th_Σ and Mod_Σ is a contravariant Galois connection.

The categories associated to the lattices of classes of Σ-algebras and families of finitary Σ-equations are related by the adjunction $\mathrm{Mod}_\Sigma \dashv \mathrm{Th}_\Sigma$, i.e., for every class $\mathcal{K}$ of Σ-algebras and every family $\mathcal{E}$ of finitary Σ-equations, we have that $\mathcal{K} \subseteq \mathrm{Mod}_\Sigma(\mathcal{E})$ iff $\mathcal{E} \subseteq \mathrm{Th}_\Sigma(\mathcal{K})$, because of the contravariance.

Definition 5. We denote by Cn_Σ the closure operator $\mathrm{Th}_\Sigma \circ \mathrm{Mod}_\Sigma$ on $\mathrm{Eq}_\mathrm{H}(\Sigma)$ and we call the Cn_Σ-closed sets *Σ-equational theories.* If $\mathcal{E}$ is a family of finitary Σ-equations and (P,Q) a finitary Σ-equation of type (w,s), then we say that (P,Q) is a *semantical consequence* of $\mathcal{E}$ if $\mathrm{Mod}_\Sigma(\mathcal{E}) \subseteq \mathrm{Mod}_\Sigma(P,Q)$, i.e., if $(P,Q) \in \mathrm{Th}_\Sigma(\mathrm{Mod}_\Sigma(\mathcal{E}))_{w,s}$.

Now we define the Hall algebras through a many-sorted equational presentation that differs from that in [5].

Definition 6. Let S be a set of sorts and V^H the $S^\star \times S$-sorted set of variables $(V_{(w,s)})_{(w,s)\in S^\star\times S}$ where, for every $(w,s) \in S^\star \times S$, $V_{(w,s)} = \{ v^{w,s}_n \mid n \in \mathbb{N} \}$. A *Hall algebra for S* is a many-sorted $(\underline{\Sigma}^\mathrm{H}, \mathcal{E}^\mathrm{H})$-algebra, where $\underline{\Sigma}^\mathrm{H}$ is $(S^\star \times S, \Sigma^\mathrm{H})$ and Σ^H is the $S^\star \times S$-sorted signature, i.e., the $(S^\star \times S)^\star \times (S^\star \times S)$-sorted set, defined as follows:

(1) For every $w \in S^\star$ and $i \in |w|$,

$$\pi^w_i \colon \lambda \longrightarrow (w, w_i),$$

where $|w|$ is the *length* of the word w and λ the *empty word* in $(S^\star \times S)^\star$.

(2) For every u, $w \in S^\star$ and $s \in S$,

$$\xi_{u,w,s}\colon ((w,s),(u,w_0),\ldots,(u,w_{|w|-1})) \longrightarrow (u,s).$$

while $\mathcal{E}^\mathrm{H}$ is the part of $\mathrm{Eq}(\Sigma^\mathrm{H}) = (\mathrm{Fr}_{\Sigma^\mathrm{H}}({\downarrow}\overline{w})^2_{(u,s)})_{(\overline{w},(u,s))\in(S^\star\times S)^\star\times(S^\star\times S)}$ defined as follows:

H1. *Projection.* For every u, $w \in S^{\star}$ and $i \in |w|$, the equation

$$\xi_{u,w,w_i}(\pi^{w}_{i}, v^{u,w_0}_{0}, \ldots, v^{u,w_{|w|-1}}_{|w|-1}) = v^{u,w_i}_{i}$$

of type $(((u,w_0),\ldots,(u,w_{|w|-1})),(u,w_i))$.

H2. *Identity.* For every $u \in S^{\star}$ and $j \in |u|$, the equation

$$\xi_{u,u,u_j}(v^{u,u_j}_{j}, \pi^{u}_{0}, \ldots, \pi^{u}_{|u|-1}) = v^{u,u_j}_{j}$$

of type $(((u,u_j)),(u,u_j))$.

H3. *Associativity.* For every u, v, $w \in S^{\star}$ and $s \in S$, the equation

$$\begin{aligned}\xi_{u,v,s}(\xi_{v,w,s}(v^{w,s}_{0}, v^{v,w_0}_{1}, \ldots, v^{v,w_{|w|-1}}_{|w|}), v^{u,v_0}_{|w|+1}, \ldots, v^{u,v_{|v|-1}}_{|w|+|v|}) = \\ \xi_{u,w,s}(v^{w,s}_{0}, \xi_{u,v,w_0}(v^{v,w_0}_{1}, v^{u,v_0}_{|w|+1}, \ldots, v^{u,v_{|v|-1}}_{|w|+|v|}), \ldots, \\ \xi_{u,v,w_{|w|-1}}(v^{v,w_{|w|-1}}_{|w|}, v^{u,v_0}_{|w|+1}, \ldots, v^{u,v_{|v|-1}}_{|w|+|v|}))\end{aligned}$$

of type $(((w,s),(v,w_0),\ldots,(v,w_{|w|-1}),(u,v_0),\ldots(u,v_{|v|-1})),(u,s))$.

Let us remark that from H3, for $w = \lambda$, we obtain the invariance of constant functions axiom in [5]:

Invariance of constant functions. For every u, $v \in S^{\star}$ and $s \in S$, we have the equation

$$\xi_{u,v,s}(\xi_{v,\lambda,s}(v^{\lambda,s}_{0}), v^{u,v_0}_{1}, \ldots, v^{u,v_{|v|-1}}_{|v|}) = \xi_{u,\lambda,s}(v^{\lambda,s}_{0})$$

of type $(((\lambda,s),(u,v_0),\ldots,(u,v_{|v|-1})),(u,s))$.

We call the formal constants π^{w}_{i} *projections*, and the formal many-sorted operations $\xi_{u,w,s}$ *substitution operators*. Moreover, we denote by $\mathbf{Alg}(\mathrm{H})$ the category of Hall algebras for S and homomorphisms between Hall algebras.

For every S-sorted set A, $\mathrm{Op}(A) = (\mathrm{Hom}_{\mathbf{Set}}(A_w, A_s))_{(w,s)\in S^{\star}\times S}$, the $S^{\star}\times S$-sorted set of many-sorted operation for A, where, for $w \in S^{\star}$, $A_w = \prod_{i\in|w|} A_{w_i}$, is endowed with a structure of Hall algebra, if we realize the projections as the true projections and the substitution operators as the generalized composition of mappings. The closed sets of this many-sorted algebra are the clones of operations and were investigated originally, for operations on ordinary sets, by Philip Hall (see e.g., [4] or [7]).

Proposition 2. *Let A be an S-sorted set and $\underline{\mathrm{Op}}(A)$ the many-sorted Σ^{H}-algebra with underlying many-sorted set $\mathrm{Op}(A)$ and many-sorted algebraic structure F defined as follows*

1. *For every $w \in S^{\star}$ and $i \in |w|$, $F_{\pi^{w}_{i}} = \mathrm{pr}^{A}_{w,i}\colon A_w \longrightarrow A_{w_i}$.*
2. *For every $u, w \in S^{\star}$ and $s \in S$, $F_{\xi_{u,w,s}}$ is defined, for every $f \in A_s^{A_w}$ and $g \in A_w^{A_u}$, as $F_{\xi_{u,w,s}}(f, g_0, \ldots, g_{|w|-1}) = f \circ \langle g_i\rangle_{i\in|w|}$, where $\langle g_i\rangle_{i\in|w|}$ is the unique mapping from A_u to A_w such that, for every $i \in |w|$, $\mathrm{pr}^{A}_{w,i}\circ\langle g_i\rangle_{i\in|w|} = g_i$.*

Then $\underline{\mathrm{Op}}(A)$ is a Hall algebra.

We remark that, as a particular case of substitution, we also have $F_{\xi_{u,\lambda,s}}$, that converts constants of type $\kappa^{a}_{\lambda,s}$ into constants of type $\kappa^{a}_{u,s}$, for $a \in A_s$ and $u \in S^{\star}$.

For every S-sorted signature Σ, $\mathrm{Pol}_{\mathrm{H}}(\Sigma) = (\mathrm{Fr}_{\Sigma}(\downarrow w)_s)_{(w,s)\in S^{\star}\times S}$ is also endowed with a structure of Hall algebra that formalizes the concept of substitution.

Proposition 3. *Let Σ be an S-sorted signature and $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$ the many-sorted Σ^{H}-algebra with underlying many-sorted set $\mathrm{Pol}_{\mathrm{H}}(\Sigma)$ and many-sorted algebraic structure F defined as follows*

1. *For every $w \in S^{\star}$ and $i \in |w|$, $F_{\pi^{w}_{i}}$ is the image under $\eta^{\downarrow w}_{w_i}$ of the variable $v^{w_i}_{i}$, where $\eta^{\downarrow w} = (\eta^{\downarrow w}_{s})_{s\in S}$ is the canonical injection of $\downarrow w$ into $\mathrm{Fr}_{\Sigma}(\downarrow w)$.*

(2) *For every $u, w \in S^\star$ and $s \in S$, $F_{\xi_{u,w,s}}$ is the mapping*

$$F_{\xi_{u,w,s}} \begin{cases} \mathrm{Fr}_\Sigma(\downarrow w)_s \times \mathrm{Fr}_\Sigma(\downarrow u)_{w_0} \times \cdots \times \mathrm{Fr}_\Sigma(\downarrow u)_{w_{|w|-1}} & \longrightarrow \mathrm{Fr}_\Sigma(\downarrow u)_s \\ (P, (Q_i \mid i \in |w|)) & \longmapsto \mathcal{Q}^\sharp_s(P) \end{cases}$$

where, for $\mathcal{Q}$ the S-sorted mapping from $\downarrow w$ to $\mathrm{Fr}_\Sigma(\downarrow u)$ canonically associated to the family $(Q_i \mid i \in |w|)$, $\mathcal{Q}^\sharp$ is the unique homomorphism from $\underline{\mathrm{Fr}}_\Sigma(\downarrow w)$ into $\underline{\mathrm{Fr}}_\Sigma(\downarrow u)$ such that $\mathcal{Q}^\sharp \circ \eta^{\downarrow w} = \mathcal{Q}$.

Then $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$ is a Hall algebra.

Now we prove that, for every $S^\star \times S$-sorted set Σ, $\underline{\mathrm{Fr}}_{\mathrm{H}}(\Sigma)$, the free Hall algebra on Σ, is isomorphic to $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$. This isomorphism together with the adjunction $\underline{\mathrm{Fr}}_{\mathrm{H}} \dashv \mathrm{G}_{\mathrm{H}}$ has as consequence that, for every S-sorted set A and S-sorted signature Σ, the sets $\mathrm{Hom}_{\mathbf{Set}^{S^\star \times S}}(\Sigma, \mathrm{Op}(A))$ and $\mathrm{Hom}_{\mathbf{Alg}(\mathrm{H})}(\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma), \underline{\mathrm{Op}}(A))$ are naturally isomorphic. Moreover, the isomorphism assigns to a many-sorted structure F on A the homomorphism of Hall algebras $\mathrm{Pd}^{(A,F)}$, defined below, from $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$ into $\underline{\mathrm{Op}}(A)$ and the inverse mapping assigns to $h\colon \underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma) \longrightarrow \underline{\mathrm{Op}}(A)$, the many-sorted algebraic structure $\mathrm{G}_{\mathrm{H}}(h) \circ \eta^\Sigma$ on A.

Definition 7. Let $\underline{A}$ be a Hall algebra and Σ an S-sorted signature. Then, for every $f\colon \Sigma \longrightarrow A$ and $u \in S^\star$, $\underline{A}^{f,u}$, the *derived many-sorted Σ-algebra of $\underline{A}$ for (f, u)*, is the many-sorted Σ-algebra with underlying S-sorted set $A^{f,u} = (A_{(u,s)})_{s \in S}$ and many-sorted algebraic structure $F^{f,u}$, defined, for every $(w, s) \in S^\star \times S$, as

$$F^{f,u}_{w,s} \begin{cases} \Sigma_{w,s} \longrightarrow \mathrm{Op}_w(A^{f,u})_s \\ \sigma \longmapsto \begin{cases} \prod_{i \in |w|} A_{(u,w_i)} \longrightarrow A_{(u,s)} \\ (a_0, \dots, a_{|w|-1}) \longmapsto \xi^{\underline{A}}_{u,w,s}(f_{(w,s)}(\sigma), a_0, \dots, a_{|w|-1}) \end{cases} \end{cases}$$

where $\mathrm{Op}_w(A^{f,u}) = (A_{(u,s)}^{\prod_{i \in |w|} A_{(u,w_i)}})_{s \in S}$ and $\mathrm{Op}_w(A^{f,u})_s = A_{(u,s)}^{\prod_{i \in |w|} A_{(u,w_i)}}$. Moreover, we denote by p^u the S-sorted mapping from $\downarrow u$ into $A^{f,u}$ defined, for every $s \in S$ and $i \in |u|$, as $p^u_s(v^s_i) = (\pi^u_i)^{\underline{A}}$, and by $(p^u)^\sharp$ the unique homomorphism from $\underline{\mathrm{Fr}}_\Sigma(\downarrow u)$ into $\underline{A}^{f,u}$ such that $(p^u)^\sharp \circ \eta^{\downarrow u} = p^u$.

Lemma 1. *Let Σ be an S-sorted signature, $\underline{A}$ a Hall algebra, $f\colon \Sigma \longrightarrow A$ and $u \in S^\star$. Then, for every $(w, s) \in S^\star \times S$, $P \in \mathrm{Fr}_\Sigma(\downarrow w)_s$ and $a \in \prod_{i \in |w|} A_{(u,w_i)}$, we have that*

$$P^{\underline{A}^{f,u}}(a_0, \dots, a_{|w|-1}) = \xi^{\underline{A}}_{u,w,s}((p^w)^\sharp_s(P), a_0, \dots, a_{|w|-1})$$

Proof. By algebraic induction on the complexity of P. □

Proposition 4. *Let Σ be an S-sorted signature. Then the Hall algebra $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$ is isomorphic to $\underline{\mathrm{Fr}}_{\mathrm{H}}(\Sigma)$.*

Proof. It is enough to prove that $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$ has the universal property of the free Hall algebra on Σ. Therefore we have to specify an $S^\star \times S$-sorted mapping h from Σ into $\mathrm{Pol}_{\mathrm{H}}(\Sigma)$ such that, for every Hall algebra $\underline{A}$ and $S^\star \times S$-sorted mapping f from Σ into A, there is a unique homomorphism $\widehat{f}$ from $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$ into $\underline{A}$ such that $\widehat{f} \circ h = f$. Let h be the $S^\star \times S$-sorted mapping defined, for every $(w, s) \in S^\star \times S$, as

$$h_{w,s} \begin{cases} \Sigma_{w,s} \longrightarrow \mathrm{Fr}_\Sigma(\downarrow w)_s \\ \sigma \longmapsto \sigma(v^s_0, \dots, v^s_{|w|-1}) \end{cases}$$

Let $\underline{A}$ be a Hall algebra, $f\colon \Sigma \longrightarrow A$ an $S^\star \times S$-sorted mapping and $\widehat{f}$ the $S^\star \times S$-sorted mapping from $\mathrm{Pol}_{\mathrm{H}}(\Sigma)$ into A defined, for every $(w, s) \in S^\star \times S$, as $\widehat{f}_{(w,s)} =$

$(p^w)^\sharp_s$. Then $\widehat{f}$ is a homomorphism of Hall algebras, because, on the one hand, for $w \in S^\star$ and $i \in |w|$ we have that

$$\begin{aligned}\widehat{f}_{(w,w_i)}((\pi^w_i)^{\underline{\mathrm{Pol}(\Sigma)}}) &= \widehat{f}_{(w,w_i)}(v^s_i)\\ &= p^w_{w_i}(v^s_i)\\ &= (\pi^w_i)^{\underline{A}}\end{aligned}$$

and, on the other hand, for $P \in \mathrm{Fr}_\Sigma(\downarrow w)_s$ and $(Q_i \mid i \in |w|) \in \mathrm{Fr}_\Sigma(\downarrow u)_w$ we have that

$$\begin{aligned}&\widehat{f}_{(u,s)}(\xi^{\underline{\mathrm{Pol}(\Sigma)}}_{u,w,s}(P, Q_0, \dots, Q_{|w|-1}))\\ &= (p^u)^\sharp_s(\mathcal{Q}^\sharp_s(P))\\ &= ((p^u)^\sharp \circ \mathcal{Q})^\sharp_s(P)\\ &= P^{\underline{A}^{f,u}}((p^u)^\sharp_{w_0}(Q_0), \dots, (p^u)^\sharp_{w_{|w|-1}}(Q_{|w|-1}))\\ &= \xi^{\underline{A}}_{u,w,s}((p^w)^\sharp_s(P), (p^u)^\sharp_{w_0}(Q_0), \dots, (p^u)^\sharp_{w_{|w|-1}}(Q_{|w|-1})) \qquad (\textit{by Lemma } 1)\\ &= \xi^{\underline{A}}_{u,w,s}(\widehat{f}_{(w,s)}(P), \widehat{f}_{(u,w_0)}(Q_0), \dots, \widehat{f}_{(u,w_{|w|-1})}(Q_{|w|-1})).\end{aligned}$$

Therefore the $S^\star \times S$-sorted mapping $\widehat{f}$ is a homomorphism. Moreover, $\widehat{f} \circ h = f$, because, for every $w \in S^\star$, $s \in S$, and $\sigma \in \Sigma_{w,s}$, we have that

$$\begin{aligned}\widehat{f}_{(w,s)}(h_{w,s}(\sigma)) &= (p^w)^\sharp_s(\sigma(v^s_0, \dots, v^s_{|w|-1}))\\ &= \sigma^{\underline{A}_w}(p^w_{w_0}(v^s_0), \dots, p^w_{w_{|w|-1}}(v^s_{|w|-1}))\\ &= \xi^{\underline{A}}_{w,w,s}(f_{(w,s)}(\sigma), (\pi^w_0)^{\underline{A}}, \dots, (\pi^w_{|w|-1})^{\underline{A}})\\ &= f_{(w,s)}(\sigma) \quad \text{(H2)}\end{aligned}$$

It is obvious that $\widehat{f}$ is the unique homomorphism such that $\widehat{f} \circ h = f$. Henceforth $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$ is isomorphic to $\underline{\mathrm{Fr}}_{\mathrm{H}}(\Sigma)$. □

Now, for every many-sorted Σ-algebra $\underline{A}$, we state the existence of a homomorphism of Hall algebras $\mathrm{Pd}^{\underline{A}}$ from $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$ into $\underline{\mathrm{Op}}(\underline{A}) = \underline{\mathrm{Op}}(A)$ such that $\mathrm{Th}_\Sigma(\underline{A})$, the finitary Σ-equational theory determined by $\underline{A}$, is precisely $\mathrm{Ker}(\mathrm{Pd}^{\underline{A}})$.

Proposition 5. *Let $\underline{A}$ be a many-sorted Σ-algebra. Then the $S^\star \times S$-sorted mapping $\mathrm{Pd}^{\underline{A}}$ from $\mathrm{Pol}_{\mathrm{H}}(\Sigma)$ into $\mathrm{Op}(\underline{A}) = \mathrm{Op}(A)$ defined as $\mathrm{Pd}^{\underline{A}} = (\mathrm{Pd}^{\underline{A}}_{(w,s)})_{(w,s)\in S^\star \times S}$ where, for every $(w,s) \in S^\star \times S$, $\mathrm{Pd}^{\underline{A}}_{(w,s)}$ is the s-th coordinate of $\mathrm{Pd}^{\underline{A}}_w = (\mathrm{Pd}^{\underline{A}}_{w,s})_{s\in S}$, the unique homomorphism from $\underline{\mathrm{Fr}_\Sigma}(\downarrow w)$ into $\underline{\mathrm{Op}}_w(\underline{A}) = \underline{A}^{A_w}$ such that $\mathrm{Pd}^{\underline{A}}_w \circ \eta^{\downarrow w} = \mathrm{p}^A_w$, where p^A_w is the S-sorted mapping from $\downarrow w$ into $\mathrm{Op}_w(\underline{A}) = A^{A_w}$ defined, for every $s \in S$ and $v^s_i \in (\downarrow w)_s$, as $\mathrm{p}^A_{w,s}(v^s_i) = \mathrm{pr}^A_{w,i}$, is a homomorphism of Hall algebras from $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$ into $\underline{\mathrm{Op}}(\underline{A})$. Moreover,* $\mathrm{Ker}(\mathrm{Pd}^{\underline{A}}) = \mathrm{Th}_\Sigma(\underline{A})$.

The last part of the Proposition just stated can be extended to classes of many-sorted Σ-algebras and, in particular, to the models of a family $\mathcal{E}$ of finitary Σ-equations. From this will follow that the operador $\mathrm{Cg}_{\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)}$ is sound relative to the operador of semantical consequence Cn_Σ.

Proposition 6. *Let $\mathcal{K}$ a class of many-sorted Σ-algebras. Then* $\mathrm{Th}_\Sigma(\mathcal{K})$ *is a congruence on* $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$.

Proof. Because $\mathrm{Th}_\Sigma(\mathcal{K})$ is $\bigcap_{\underline{A}\in\mathcal{K}} \mathrm{Ker}(\mathrm{Pd}^{\underline{A}}) \in \mathrm{Cgr}(\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma))$. □

Corollary 1 (Soundness Theorem)**.** *Let Σ be an S-sorted signature. Then we have that* $\mathrm{Cg}_{\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)} \leq \mathrm{Cn}_\Sigma$.

Proof. Let $\mathcal{E}$ be a part of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$. By definition $\mathrm{Cn}_{\Sigma}(\mathcal{E}) = \mathrm{Th}_{\Sigma}(\mathrm{Mod}_{\Sigma}(\mathcal{E}))$, that is a congruence on $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$ and contains $\mathcal{E}$, therefore $\mathrm{Cn}_{\Sigma}(\mathcal{E})$ contains $\mathrm{Cg}_{\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)}(\mathcal{E})$. □

The congruence generated in $\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)$ by a family of finitary Σ-equations $\mathcal{E}$ can be characterized as follows.

Proposition 7. *Let $\mathcal{E}$ be a part of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$. Then $\mathrm{Cg}_{\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)}(\mathcal{E})$ is the smallest part $\overline{\mathcal{E}}$ of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ that contains $\mathcal{E}$ and is such that, for every $u, w \in S^{\star}$ and $s \in S$, satisfies the following conditions:*

1. Reflexivity. *For every $P \in \mathrm{Pol}_{\mathrm{H}}(\Sigma)_{w,s}$, $(P, P) \in \overline{\mathcal{E}}_{w,s}$.*
2. Symmetry. *For every P, $Q \in \mathrm{Pol}_{\mathrm{H}}(\Sigma)_{w,s}$, if $(P, Q) \in \overline{\mathcal{E}}_{w,s}$, then $(Q, P) \in \overline{\mathcal{E}}_{w,s}$.*
3. Transitivity. *For every P, Q, $R \in \mathrm{Pol}_{\mathrm{H}}(\Sigma)_{w,s}$, if (P, Q), $(Q, R) \in \overline{\mathcal{E}}_{w,s}$, then $(P, R) \in \overline{\mathcal{E}}_{w,s}$.*
4. Substitutivity. *For every $(M_i \mid i \in |w|), (N_i \mid i \in |w|) \in \prod_{i \in |w|} \mathrm{Pol}_{\mathrm{H}}(\Sigma)_{u,w_i}$ such that, for every $i \in |w|$, $(M_i, N_i) \in \overline{\mathcal{E}}_{u,w_i}$, and $(P, Q) \in \overline{\mathcal{E}}_{w,s}$,*

$$(\xi_{u,w,s}(P, M_0, \dots, M_{|w|-1}), \xi_{u,w,s}(Q, N_0, \dots, N_{|w|-1})) \in \overline{\mathcal{E}}_{u,s}.$$

□

Let us remark that in the Proposition just stated, the substitutivity condition for $w = \lambda$ demands that if $(P, Q) \in \overline{\mathcal{E}}_{\lambda,s}$ then, for every $u \in S^{\star}$, $(P, Q) \in \overline{\mathcal{E}}_{u,s}$.

Proposition 8. *Let $\mathcal{E}$ be a part of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ and $\sigma \in \Sigma_{w,s}$. If, for every $i \in |w|$, $(P_i, Q_i) \in \overline{\mathcal{E}}_{w,w_i}$, then $(\sigma(P_0, \dots, P_{|w|-1}), \sigma(Q_0, \dots, Q_{|w|-1})) \in \overline{\mathcal{E}}_{w,s}$.*

Proof. By reflexivity $(\sigma(v_0, \dots, v_{|w|-1}), \sigma(v_0, \dots, v_{|w|-1})) \in \overline{\mathcal{E}}_{w,s}$ hence, by substitutivity, $(\sigma(P_0, \dots, P_{|w|-1}), \sigma(Q_0, \dots, Q_{|w|-1})) \in \overline{\mathcal{E}}_{w,s}$ □

Proposition 9. *Let $\mathcal{E}$ be a part of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ and $(w, s) \in S^{\star} \times S$. If $(P, Q) \in \overline{\mathcal{E}}_{w,s}$ and f is an endomorphism of $\underline{\mathrm{Fr}}_{\Sigma}(\downarrow w)$, then $(f_s(P), f_s(Q)) \in \overline{\mathcal{E}}_{w,s}$.*

Proof. For every $i \in |w|$, the equation $(f_{w_i}(v_i), f_{w_i}(v_i))$ is in $\overline{\mathcal{E}}_{w,w_i}$. By substitutivity, we have that

$$(\xi_{w,w,s}(P, f_{w_0}(v_0), \dots, f_{w_{|w|-1}}(v_{|w|-1})), \xi_{w,w,s}(Q, f_{w_0}(v_0), \dots, f_{w_{|w|-1}}(v_{|w|-1})))$$

is in $\overline{\mathcal{E}}_{w,s}$, hence $(f_s(P), f_s(Q)) \in \overline{\mathcal{E}}_{w,s}$. □

Corollary 2. *Let $\mathcal{E}$ be a part of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ and $w \in S^{\star}$. Then $\overline{\mathcal{E}}_w = (\overline{\mathcal{E}}_{w,s})_{s \in S}$ is a fully invariant congruence on $\underline{\mathrm{Fr}}_{\Sigma}(\downarrow w)$.*

We remark that the congruence $\overline{\mathcal{E}}_w$ contains $\mathrm{Cg}^{\mathrm{fi}}_{\underline{\mathrm{Fr}}_{\Sigma}(\downarrow w)}(\mathcal{E}_w)$, the fully invariant congruence generated by $\mathcal{E}_w = (\mathcal{E}_{w,s})_{s \in S}$ and, in general, the containment is strict.

Proposition 10. *Let $\mathcal{E}$ be a part of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ and $w \in S^{\star}$. Then $\underline{\mathrm{Fr}}_{\Sigma}(\downarrow w)/\overline{\mathcal{E}}_w$ is a model of $\mathcal{E}$.*

Proposition 11 (Adequacy Theorem)**.** *Let Σ be an S-sorted signature. Then we have that $\mathrm{Cn}_{\Sigma} \leq \mathrm{Cg}_{\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)}$.*

Proof. Let $\mathcal{E}$ be a part of $\mathrm{Eq_H}(\Sigma)$. If $(P,Q) \in \mathrm{Cn}_\Sigma(\mathcal{E})_{w,s}$, then, because $\underline{\mathrm{Fr}}_\Sigma(\downarrow w)/\overline{\mathcal{E}}_w$ is a model of $\mathcal{E}$, $P^{\underline{\mathrm{Fr}}_\Sigma(\downarrow w)/\overline{\mathcal{E}}_w} = Q^{\underline{\mathrm{Fr}}_\Sigma(\downarrow w)/\overline{\mathcal{E}}_w}$. Hence

$$\begin{aligned}
[P] &= [\xi_{w,w,s}(P, \pi^w_0, \dots, \pi^w_{|w|-1})] \\
&= [P^{\underline{\mathrm{Fr}}_\Sigma(\downarrow w)}(v_0, \dots, v_{|w|-1})] \\
&= P^{\underline{\mathrm{Fr}}_\Sigma(\downarrow w)/\overline{\mathcal{E}}_w}([v_0], \dots, [v_{|w|-1}]) \\
&= Q^{\underline{\mathrm{Fr}}_\Sigma(\downarrow w)/\overline{\mathcal{E}}_w}([v_0], \dots, [v_{|w|-1}]) \\
&= [Q^{\underline{\mathrm{Fr}}_\Sigma(\downarrow w)}(v_0, \dots, v_{|w|-1})] \\
&= [\xi_{w,w,s}(Q, \pi^w_0, \dots, \pi^w_{|w|-1})] \\
&= [Q],
\end{aligned}$$

and $(P,Q) \in \mathrm{Cg}_{\underline{\mathrm{Pol}}_\mathrm{H}(\Sigma)}(\mathcal{E})_{w,s}$. □

Corollary 3 (Completeness Theorem of Goguen-Meseguer)**.** *Let Σ be an S-sorted signature. Then we have that* $\mathrm{Cg}_{\underline{\mathrm{Pol}}_\mathrm{H}(\Sigma)} = \mathrm{Cn}_\Sigma$.

The Completeness Theorem of Goguen-Meseguer allows us to obtain a calculus of finitary Σ-equations, i.e., a calculus on sets of variables of the form $\downarrow w$, for $w \in S^\star$, or, what amounts to the same, on finite subsets of V. Before we state the finitary Σ-equational inference rules we agree that $(P,Q) : (X,s)$ means that the finitary Σ-ecuación (P,Q) is of type (X,s), i.e., that $P,Q \in \mathrm{Fr}_\Sigma(X)_s$, in addition if $P \in \mathrm{Fr}_\Sigma(X)_s$ and $\mathcal{P} = (P_s)_{s\in S}\colon X \longrightarrow \mathrm{Fr}_\Sigma(Y)$, then $P(x/P_{s,x})_{s\in S,\, x\in X_s}$ is $\mathcal{P}^\sharp_s(P)$.

Proposition 12 (Inference Rules)**.** *The following finitary Σ-equational inference rules determine a closure operator on $\mathrm{Eq_H}(\Sigma)$ that is identical to the closure operator Cn_Σ.*

R1 Reflexivity. *For all $P \in \mathrm{Fr}_\Sigma(X)_s$, $(P,P) \in \overline{\mathcal{E}}_{X,s}$, or diagrammatically*

$$\frac{}{(P,P) : (X,s)} \quad P \in \mathrm{Fr}_\Sigma(X)_s$$

R2 Symmetry. *For all $P,Q \in \mathrm{Fr}_\Sigma(X)_s$, if $(P,Q) \in \overline{\mathcal{E}}_{X,s}$, then $(Q,P) \in \overline{\mathcal{E}}_{X,s}$, or diagrammatically*

$$\frac{(P,Q) : (X,s)}{(Q,P) : (X,s)}$$

R3 Transitivity. *For all $P,Q,R \in \mathrm{Fr}_\Sigma(X)_s$, if $(P,Q) \in \overline{\mathcal{E}}_{X,s}$ and $(Q,R) \in \overline{\mathcal{E}}_{X,s}$, then $(P,R) \in \overline{\mathcal{E}}_{X,s}$ or diagrammatically*

$$\frac{(P,Q) : (X,s) \qquad (Q,R) : (X,s)}{(P,R) : (X,s)}$$

R4 Generalized substitutivity. *For all $(P,Q) \in \overline{\mathcal{E}}_{X,s}$ and $\mathcal{P}, \mathcal{Q}\colon X \longrightarrow \mathrm{Fr}_\Sigma(Y)$ such that, for every $s \in S$, $x \in X_s$, $(P_{s,x}, Q_{s,x}) \in \overline{\mathcal{E}}_{Y,s}$,*

$$(\xi_{Y,X,s}(P, (P_{s,x})_{s\in S,\, x\in X_s}), \xi_{Y,X,s}(Q, (Q_{s,x})_{s\in S,\, x\in X_s})) \in \overline{\mathcal{E}}_{Y,s},$$

or diagrammatically

$$\frac{(P,Q) : (X,s) \qquad ((P_{s,x}, Q_{s,x}) : (Y,s))_{s\in S,\, x\in X_s}}{(P(x/P_{s,x})_{s\in S,\, x\in X_s}, Q(x/Q_{s,x})_{s\in S,\, x\in X_s}) : (Y,s)}$$

Proof. Because the finitary Σ-equational inference rules are the translation of the conditions in Proposition 7. □

Proposition 13. *The inference rule R4 is equivalent, assuming R1, to the following inference rule*

R4′ Substitutivity.

$$\frac{(P,Q):(X,s) \qquad (P',Q'):(Y,t)}{(P(x/P'),Q(x/Q')):((X-\delta^{t,x})\cup Y,s)} \quad x\in X_t\ [\delta^{t,x}_t=\{x\},\ \delta^{t,x}_s=\varnothing,\ \mathit{if}\ s\neq t]$$

Proof. We begin by proving that R4 implies R4′. If $(P,Q):(X,s)$ and $(P',Q'):(Y,t)$ are deducible and $x\in X_t$, then also are deducible, by reflexivity, the finitary Σ-equations in the family $((P''_{s,x},Q''_{s,x}):((X-\delta^{t,x})\cup Y,s))_{s\in S,\,x\in X_s}$, where $P''_{t,x}=P'$, $Q''_{t,x}=Q'$, and otherwise $P''_{s,y}=Q''_{s,y}=y$. Then, by generalized substitutivity, $(P(x/P'),Q(x/Q')):((X-\delta^{t,x})\cup Y,s)$ is deducible, because $P(x/P')=(P(x/P''_{s,x})_{s\in S,\,x\in X_s}$ and $Q(x/Q')=Q(x/P''_{s,x})_{s\in S,\,x\in X_s}$.

Reciprocally, R4′ implies R4, by reiterating the application of R4′ $\mathrm{card}(\coprod X)$-times. □

In some presentations of many-sorted equational logic, e.g., in [5], are introduced two additional inference rules that allow the adjunction and suppresion of variables, under some conditions. But as we will prove below both rules are derived rules, relative to the system of rules R1 to R4.

Definition 8 (Abstraction and concretion)**.**

R5 *Abstraction.*

$$\frac{(P,Q):(X,s)}{(P,Q):(X\cup\delta^{t,x},s)} \quad x\in V_t-X_t$$

R6 *Concretion.*

$$\frac{(P,Q):(X,s)}{(P,Q):(X-\delta^{t,x},s)} \quad x\in X_t,\ x\notin \mathrm{var}(P,Q),\ \mathrm{Fr}_\Sigma((\varnothing)_{s\in S})_t\neq\varnothing.$$

Proposition 14. *The abstraction and concretion rules are derived rules.*

Proof. Abstraction is a derived rule. Let $y\in V_s$ be such that $y\notin X_s$. Then, by reflexivity, the finitary Σ-equation $(y,y):(\delta^{s,y}\cup\delta^{t,x},s)$ is deducible. Hence, by substitutivity, the finitary Σ-equation

$$(y(y/P),y(y/Q)):(((\delta^{s,y}\cup\delta^{t,x})-\delta^{s,y})\cup X,s)$$

that is identical to $(P,Q):(X\cup\delta^{t,x},s)$, is also deducible. As a particular case we have that if $(P,Q):((\varnothing)_{s\in S},s)$ is deducible, then $(P,Q):(\delta^{t,x},s)$ is also deducible.

Concretion is a derived rule. Since $\mathrm{Fr}_\Sigma((\varnothing)_{s\in S})_t\neq\varnothing$ let us choose an $R\in\mathrm{Fr}_\Sigma((\varnothing)_{s\in S})_t$. Then, by reflexivity, the finitary Σ-equation $(R,R):((\varnothing)_{s\in S},t)$ is deducible. Hence, by substitutivity, $(P(x/R),Q(x/R)):((X-\delta^{t,x})\cup(\varnothing)_{s\in S},s)$ is also deducible and, because $x\notin\mathrm{var}(P,Q)$, $(P,Q):(X-\delta^{t,x},s)$ is deducible. □

Definition 9 (Replacement rule)**.**

R7 *Replacement.*

$$\frac{(P^i,Q^i):(X,w_i)}{(\sigma(P_0,\ldots,P_{|w|-1}),\sigma(Q_0,\ldots,Q_{|w|-1})):(X,s)} \quad \sigma\in\Sigma_{w,s}$$

Proposition 15. *The replacement rule is a derived rule.*

Proof. By reflexivity, $(\sigma(v_0,\ldots,v_{|w|-1}),\sigma(v_0,\ldots,v_{|w|-1})):(\downarrow w,s)$ is deducible. Now, by reiterating substitutivity $|w|$-times, we obtain the desired finitary Σ-equation. □

Everything we have made up to now can be extended to the case of locally finitary Σ-equations, i.e., pairs of mappings from δ^s to $\mathrm{Fr}_\Sigma(X)$, for some $s\in S$ and $X\in\mathrm{Sub}_{\mathrm{lf}}(V)=\{\,X\subseteq V\mid \forall s\in S\,(X_s\text{ is finite})\,\}$, we only have to change the structural operations of the Hall algebras to locally finitary operations. Moreover, the equational calculus has the same inference rules R1–R4, but generalized to locally finite S-sorted sets of variables. However, the rule of substitution is not more equivalent to the generalized rule of substitution. Finally, the rules of abstraction and concretion for this case are the following.

Definition 10.

R5′ *Generalized abstraction.*

$$\frac{(P,Q):(X,s)}{(P,Q):(X\cup Y,s)}$$

R6′ *Generalized concretion.*

$$\frac{(P,Q):(X,s)}{(P,Q):(X-Y,s)}\quad Y\cap \mathrm{var}(P,Q)=\varnothing,\ \mathrm{supp}(Y)\subseteq \mathrm{supp}(\mathrm{Fr}_{\Sigma}((\varnothing)_{s\in S}))$$

Where, for an S-sorted set Z, $\mathrm{supp}(Z)$, the *support* of Z, is $\{\, s\in S \mid Z_s\neq \varnothing \,\}$.

The Completeness Theorem can also be proved alternatively by using instead of the Hall algebras the Bénabou algebras. This is interesting because, on the one hand, the category of Bénabou algebras is isomorphic to the category of Bénabou theories defined in [1] and, on the other hand, the Bénabou algebras even having an equational presentation radically different from that of the Hall algebras, are equivalent to them, i.e., the respective categories are equivalent. In order to accomplish such an alternative proof we begin by defining the Bénabou algebras.

Definition 11. Let S be a set of sorts and V^{B} the $S^{\star}\times S^{\star}$-sorted set of variables $(V_{(u,w)})_{(u,w)\in S^{\star}\times S^{\star}}$ where, for every $(u,w)\in S^{\star}\times S^{\star}$, $V_{(u,w)}=\{\, v_n^{u,w} \mid n\in\mathbb{N}\,\}$. A *Bénabou algebra for S* is a many-sorted $(\underline{\Sigma}^{\mathrm{B}},\mathcal{E}^{\mathrm{B}})$-algebra, where $\underline{\Sigma}^{\mathrm{B}}$ is $(S^{\star}\times S^{\star},\Sigma^{\mathrm{B}})$ and Σ^{B} is the $(S^{\star})^2$-sorted signature, i.e., the $(S^{\star}\times S^{\star})^{\star}\times(S^{\star}\times S^{\star})$-sorted set, defined as follows:

(1) For every $w\in S^{\star}$ and $i\in|w|$,
$$\pi_i^w\colon \lambda\longrightarrow (w,(w_i)).$$
(2) For every $u,\ w\in S^{\star}$,
$$\langle\,\rangle_{u,w}\colon ((u,(w_0)),\ldots,(u,(w_{|w|-1})))\longrightarrow (u,w).$$
(3) For every $u,\ x,\ w\in S^{\star}$,
$$\circ_{u,x,w}\colon ((u,x),(x,w))\longrightarrow (u,w).$$

while $\mathcal{E}^{\mathrm{B}}$ is the part of $\mathrm{Eq}(\Sigma^{\mathrm{B}})=(\mathrm{Fr}_{\Sigma^{\mathrm{B}}}(\downarrow\overline{w})^2_{(u,x)})_{(\overline{w},(u,x))\in(S^{\star}\times S^{\star})^{\star}\times(S^{\star}\times S^{\star})}$ defined as follows:

B1. For every $u,\ w\in S^{\star}$ and $i\in|w|$, the equation
$$\pi_i^w\circ_{u,w,(w_i)}\langle v_0^{u,(w_0)},\ldots,v_{|w|-1}^{u,(w_{|w|-1})}\rangle_{u,w}=v_i^{u,(w_i)}$$
of type $(((u,(w_0)),\ldots,(u,(w_{|w|-1}))),(u,(w_i)))$.

B2. For every u and $w\in S^{\star}$, the equation
$$v_0^{u,w}\circ_{u,u,w}\langle \pi_0^u,\ldots,\pi_{|u|-1}^u\rangle_{u,u}=v_0^{u,w}$$
of type $(((u,w)),(u,w))$.

B3. For every u and $w\in S^{\star}$, the equation
$$\langle \pi_0^w\circ_{u,w,w_0} v_0^{u,w},\ldots,\pi_{|w|-1}^w\circ_{u,w,w_{|w|-1}} v_0^{u,w}\rangle_{u,w}=v_0^{u,w}$$
of type $(((u,w)),(u,w))$.

B4. For every $w\in S^{\star}$, the equation
$$\langle \pi_0^w\rangle_{w,(w_0)}=\pi_0^w$$
of type $(((w,(w_0))),(w,(w_0)))$.

B5. For every u, x, w, $y \in S^\star$, the equation
$$v_0^{w,y} \circ_{u,w,y} (v_1^{x,w} \circ_{u,x,w} v_2^{u,x}) = (v_0^{w,y} \circ_{x,w,y} v_1^{x,w}) \circ_{u,x,y} v_2^{u,x}$$
of type $(((w,y),(x,w),(u,x)),(u,y))$.

where $v_n^{u,w}$ is the n-th variable of type (u,w), $Q \circ_{u,x,w} P$ is $\circ_{u,x,w}(P,Q)$, and $\langle P_0,\dots,P_{|w|-1}\rangle_{u,w}$ is $\langle\,\rangle_{u,w}(P_0,\dots,P_{|w|-1})$. We will write $\circ$ instead of $\circ_{u,x,w}$ and $\langle\dots\rangle$ instead of $\langle\dots\rangle_{u,w}$, if there is not risk of misunderstanding. Moreover, we denote by $\mathbf{Alg}(\mathrm{B})$ the category of Bénabou algebras and homomorphisms between them.

For every S-sorted set A, $\mathrm{Op}_{\mathrm{B}_S}(A) = (\mathrm{Hom}_{\mathbf{Set}}(A_w, A_u))_{(w,u)\in S^\star\times S^\star}$ is endowed with a structure of Bénabou algebra.

For every S-sorted signature Σ, $\mathrm{Pol}_{\mathrm{B}}(\Sigma) = (\mathrm{Hom}_{\mathbf{Set}^S}(\downarrow u, \mathrm{Fr}_{\underline{\Sigma}}(\downarrow w))_{(w,u)\in S^\star\times S^\star}$ is endowed with a structure of Bénabou algebra if the projections π_i^w are interpreted as the variables $v_i^{w_i}$, the operators $\langle\rangle_{u,w}$ as the isomorphisms that transform w-families of formal $\underline{\Sigma}$-polynomials on $\downarrow u$ into S-sorted mappings from $\downarrow w$ to $\mathrm{Fr}_{\underline{\Sigma}}(\downarrow u)$, and the operators $\circ_{u,x,w}$ as the substitutions for families of formal polynomials, that to families $\mathcal{P} \in \mathrm{Fr}_{\underline{\Sigma}}(\downarrow u)_x$ and $\mathcal{Q} \in \mathrm{Fr}_{\underline{\Sigma}}(\downarrow x)_w$, assign $\mathcal{Q} \circ_{u,x,w} \mathcal{P} = \mathcal{P}^\sharp \circ \mathcal{Q} \in \mathrm{Fr}_{\underline{\Sigma}}(\downarrow u)_w$.

Now, once defined the many-sorted algebraic theories of Bénabou and the morphisms between them, we prove that the category of Bénabou algebras is isomorphic to the category of many-sorted algebraic theories of Bénabou.

Definition 12. Let S be a set of sorts.

(1) A *many-sorted algebraic theory of Bénabou for* S or, to simplify, a *Bénabou theory for* S, is a pair $\underline{\mathbf{L}} = (\mathbf{L}, p)$ where $\mathbf{L}$ is a category with objects the words on S and p a family $(p^w)_{w\in S^\star}$ such that, for every word $w \in S^\star$, p^w is a family $(p_i^w\colon w \longrightarrow (w_i))_{i\in|w|}$ of morphisms in $\mathbf{L}$, that we call the *projections* for w, such that (w, p^w) is a product in $\mathbf{L}$ of the family $((w_i))_{i\in|w|}$.

(2) Let $\underline{\mathbf{L}}$ and $\underline{\mathbf{L}}'$ be two Bénabou theories for S. A *morphism on* $\underline{\mathbf{L}}$ *to* $\underline{\mathbf{L}}'$ is a functor F on $\mathbf{L}$ to $\mathbf{L}'$ such that the object mapping of F is the identity and the morphism mapping of F preserves the projections, i.e., for every $w \in S^\star$ and $i \in |w|$, $F((p_i^w)^{\underline{\mathbf{L}}}) = (p_i^w)^{\underline{\mathbf{L}}'}$.

Proposition 16. *Let S be a set of sorts. The Bénabou theories for S together with the morphisms between them determine a category* $\mathbf{BTh}(S)$.

Proposition 17. *Let $\underline{L}$ be a Bénabou algebra for S. Then $\underline{\mathbf{L}} = (\mathbf{L}, \pi)$ with $\mathbf{L}$ the category defined as follows*

(1) $\mathrm{Ob}(\mathbf{L}) = S^\star$ *and* $\mathbf{L}(u,w) = L_{u,w}$.

(2) *For every* $w \in S^\star$, $\mathrm{id}_w = \langle(\pi_i^w)^{\underline{L}} \mid i \in |w|\rangle_{w,w}$.

(3) *If $P\colon u \longrightarrow w$, $Q\colon w \longrightarrow x$, then the composition of P and Q is* $\circ^{\underline{L}}_{u,w,x}(P,Q)$.

and π the mapping defined, for every $w \in S^\star$, as $\pi^w = ((\pi_i^w)^{\underline{L}})_{i\in|w|}$, is a Bénabou theory for S.

Proof. We begin by proving that, for every $x \in S^\star$, id_x is an identity in $\mathbf{L}$. Let $P\colon u \longrightarrow x$ and $Q\colon x \longrightarrow w$ be morphisms in $\mathbf{L}$. Then we have that

$$\begin{aligned} P &= \langle(\pi_i^x)^{\underline{L}} \circ P \mid i \in |x|\rangle && \text{(by B3)} \\ &= \langle(\pi_i^x)^{\underline{L}} \circ (\langle(\pi_i^x)^{\underline{L}} \mid i \in |x|\rangle \circ P) \mid i \in |x|\rangle && \text{(by B1 and B5)} \\ &= \langle(\pi_i^x)^{\underline{L}} \mid i \in |x|\rangle \circ P && \text{(by B3)} \end{aligned}$$

$$Q = Q \circ \langle(\pi_i^x)^{\underline{L}} \mid i \in |x|\rangle \qquad \text{(by B2)}$$

The composition is associative by B5.

Now we prove that, for every $w \in S^\star$, $(w, ((\pi_i^w)^{\underline{L}})_{i\in|w|})$ is a product in $\mathbf{L}$ of the family $((w_i)_{i\in|w|})$. If $(P_i\colon x \longrightarrow w_i)_{i\in|w|}$ is a family of morphisms, then we have that

$$(\pi_i^w)^{\underline{L}} \circ \langle P_i \mid i \in |w|\rangle = P_i \quad \text{(by B1)}$$

Moreover, if $Q\colon x \longrightarrow w$ is such that $(\pi_i^w)^{\underline{L}} \circ Q = P_i$, then

$$\begin{aligned} Q &= \langle (\pi_i^w)^{\underline{L}} \circ Q \mid i \in |w|\rangle \quad \text{(by B3)} \\ &= \langle P_i \mid i \in |w|\rangle \end{aligned}$$

□

Proposition 18. *Let $\underline{\mathbf{L}} = (\mathbf{L}, p)$ be a Bénabou theory for S. Then the family*

$$(L_{w,u})_{(w,u)\in(S^\star)^2} = (\mathbf{L}(w,u))_{(w,u)\in(S^\star)^2}$$

together with, for every $w \in S^\star$ and $i \in |w|$, $\pi_i^w = p_i^w$, for every $u, w \in S^\star$, $\langle\rangle_{u,w}$ the mapping on $\prod_{i\in|w|} \mathbf{L}(u, w_i)$ to $\mathbf{L}(u,w)$ obtained by the universal property of the product for w, and, for every $u, x, w \in S^\star$, $\circ_{u,w,x}$ the composition in $\mathbf{L}$, is a Bénabou algebra $\underline{L}$.

Proposition 19. *The categories $\mathbf{Alg}(\mathrm{B}_S)$ and $\mathbf{BTh}(S)$ are isomorphic.*

Proof. Let T be the functor on $\mathbf{Alg}(\mathrm{B}_S)$ to $\mathbf{BTh}(S)$ that to a Bénabou algebra $\underline{L}$ assigns the Bénabou theory $(\mathbf{L}, \pi^{\underline{L}})$, and to a morphism of Bénabou algebras $f\colon \underline{L} \longrightarrow \underline{K}$ assigns the morphism of Bénabou theories $T(f)$ that to $P\colon w \longrightarrow u$ associates $f_{w,u}(P)\colon w \longrightarrow u$.

Let A be the functor on $\mathbf{BTh}(S)$ to $\mathbf{Alg}(\mathrm{B}_S)$ that to a Bénabou theory $\underline{\mathbf{L}} = (\mathbf{L}, p)$ assigns the Bénabou algebra corresponding to $\underline{L}$ and to a morphism of Bénabou theories $F\colon \underline{\mathbf{L}} \longrightarrow \underline{\mathbf{L}}'$ assigns the morphism of Bénabou algebras, that for $u, w \in S^\star$, is the bi-restriction of F to $\mathbf{L}(u,w)$ and $\mathbf{L}'(u,w)$.

The functors T and A are mutually inverses, therefore the categories $\mathbf{Alg}(\mathrm{B}_S)$ and $\mathbf{BTh}(S)$ are isomorphic. □

Now we state the equivalence between the categories of Hall and Bénabou algebras.

Proposition 20. *The categories $\mathbf{Alg}(\mathrm{H})$ and $\mathbf{Alg}(\mathrm{B})$ are equivalent.*

Proof. Let $\underline{\mathrm{B}}\colon \mathbf{Alg}(\mathrm{H}) \longrightarrow \mathbf{Alg}(\mathrm{B})$ be the functor that to a Hall algebra $\underline{A}$ assigns the Bénabou algebra $\underline{\mathrm{B}}(\underline{A})$ that has as underlying $S^\star \times S^\star$-sorted set $\mathrm{B}(A) = ((A_w)_u)_{(w,u)\in(S^\star)^2}$, where $A_w = (A_{w,s})_{s\in S}$ and $(A_w)_u = \prod_{i\in|u|} A_{w,u_i}$, and as algebraic structure that defined as

$$\begin{aligned} (\pi_i^w)^{\underline{\mathrm{B}}(\underline{A})} &= ((\pi_i^w)^{\underline{A}}), \\ \langle (a_0), \ldots, (a_{|w|-1})\rangle_{u,w}^{\underline{\mathrm{B}}(\underline{A})} &= (\xi_{u,w,w_0}^{\underline{A}}(\pi_0^w, a_0, \ldots, a_{|w|-1}), \ldots \\ &\qquad \xi_{u,w,w_{|w|-1}}^{\underline{A}}(\pi_{|w|-1}^w, a_0, \ldots, a_{|w|-1})), \\ \circ_{u,x,w}^{\underline{\mathrm{B}}(\underline{A})}(a,b) &= (\xi_{u,x,w_0}^{\underline{A}}(b_0, a_0, \ldots, a_{|x|-1}), \ldots \\ &\qquad \xi_{u,x,w_{|w|-1}}^{\underline{A}}(b_0, a_0, \ldots, a_{|x|-1})); \end{aligned}$$

and to an homomorphism $f\colon \underline{A} \longrightarrow \underline{B}$ of Hall algebras assigns the homomorphism $\mathrm{B}(f) = ((f_w)_u)_{(w,u)\in(S^\star)^2}$, defined for $(a_0, \ldots, a_{|u|-1})$ in $(A_w)_u$ as

$$(a_0, \ldots, a_{|u|-1}) \longmapsto (f_{w,u_0}(a_0), \ldots, f_{w,u_{|u-1|}}(a_{|u|-1}))).$$

Reciprocally, let $\underline{\mathrm{H}}\colon \mathbf{Alg}(\mathrm{B}) \longrightarrow \mathbf{Alg}(\mathrm{H})$ be the functor that to a Bénabou algebra $\underline{A}$ assigns the Hall algebra $\underline{\mathrm{H}}(\underline{A})$ that has $\mathrm{H}(A) = (A_{w,(s)})_{(w,s)\in S^\star\times S}$ as underlying $S^\star \times S$-sorted set, and as algebraic structure that defined as

$$(\pi^w_i)^{\underline{\mathrm{H}}(\underline{A})} = (\pi^w_i)^{\underline{A}},$$
$$\xi^{\underline{\mathrm{H}}(\underline{A})}_{u,w,s}(a_0, a_1, \ldots, a_{|w|}) = a_0 \circ_{u,w,s} \langle a_1, \ldots, a_{|w|}\rangle_{u,w};$$

and to an homomorphism $f\colon \underline{A} \longrightarrow \underline{B}$ of Bénabou algebras assigns the bi-restriction of f to $\mathrm{B}(A)$ and $\mathrm{B}(B)$.

Next, for a Bénabou algebra $\underline{A}$, we prove that $\underline{A}$ and $\underline{\mathrm{B}}\,\underline{\mathrm{H}}(\underline{A})$ are isomorphic. Let $f\colon \underline{A} \longrightarrow \underline{\mathrm{B}}\,\underline{\mathrm{H}}(A)$ be the $S^\star \times S^\star$-sorted mapping defined, for $(u,w) \in S^\star \times S^\star$ and $a \in A_{u,w}$, as

$$a \mapsto ((\pi^w_0)^{\underline{A}} \circ a, \ldots, (\pi^w_{|w|-1})^{\underline{A}} \circ a).$$

The definition is sound because, for $a \in A_{u,w}$, we have that $(\pi^w_i)^{\underline{A}} \circ a \in \mathrm{H}(A)_{u,w_i}$, hence $((\pi^w_0)^{\underline{A}} \circ a, \ldots, (\pi^w_{|w|-1})^{\underline{A}} \circ a) \in \mathrm{B}\,\mathrm{H}(A)_{u,w}$. Thus defined it is easy to prove that f is a homomorphism.

Reciprocally, let $g\colon \underline{\mathrm{B}}\,\underline{\mathrm{H}}(\underline{A}) \longrightarrow \underline{A}$ be the $S^\star \times S^\star$-sorted mapping defined, for $(u,w) \in S^\star \times S^\star$ and $b \in \mathrm{B}\,\mathrm{H}(A)$, as

$$b \mapsto \langle b_0, \ldots, b_{|w|-1}\rangle^{\underline{A}}$$

The definition is sound because, for $b = (b_0, \ldots, b_{|w|-1}) \in \mathrm{B}\,\mathrm{H}(A)$, we have that $b_i \in \mathrm{H}(A)_{u,w_i}$, hence $b_i \in A_{u,(w_i)}$, therefore $\langle b_0, \ldots, b_{|w|-1}\rangle^{\underline{A}} \in A_{u,w}$. Thus defined it is easy to prove that g is a homomorphism.

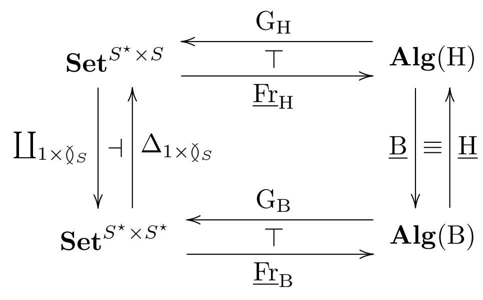
Now we prove that the homomorphisms f and g are such that $g \circ f = \mathrm{id}_{\underline{A}}$ and $f \circ g = \mathrm{id}_{\underline{\mathrm{B}}\,\underline{\mathrm{H}}(\underline{A})}$. On the one hand, if $a \in A_{u,w}$, then $\langle (\pi^w_0)^{\underline{A}} \circ a, \ldots, (\pi^w_{|w|-1})^{\underline{A}} \circ a\rangle = a$ by B3, hence $g \circ f = \mathrm{id}_{\underline{A}}$. On the other hand, if $b \in \mathrm{B}\,\mathrm{H}(A)$, $f_{u,w} \circ g_{u,w}(b)$ is the mapping

$$\begin{aligned}
b \mapsto{}& \langle b_0, \ldots, b_{|w|-1}\rangle^{\underline{A}}_{u,w}\\
\mapsto{}& ((\pi^w_0)^{\underline{\mathrm{B}}\,\underline{\mathrm{H}}(\underline{A})} \circ \langle b_0, \ldots, b_{|w|-1}\rangle^{\underline{A}}_{u,w}, \ldots, (\pi^w_{|w|-1})^{\underline{\mathrm{B}}\,\underline{\mathrm{H}}(\underline{A})} \circ \langle b_0, \ldots, b_{|w|-1}\rangle^{\underline{A}}_{u,w})\\
={}& ((\pi^w_0)^{\underline{A}} \circ \langle b_0, \ldots, b_{|w|-1}\rangle^{\underline{A}}_{u,w}, \ldots, (\pi^w_{|w|-1})^{\underline{A}} \circ \langle b_0, \ldots, b_{|w|-1}\rangle^{\underline{A}}_{u,w})\\
={}& \langle b_0, \ldots, b_{|w|-1}\rangle^{\underline{A}}_{u,w}
\end{aligned}$$

where the last step is justified by the axiom B1, hence $f \circ g = \mathrm{id}_{\underline{\mathrm{B}}\,\underline{\mathrm{H}}(\underline{A})}$.

Finally, for a Hall algebra $\underline{A}$ we have that $\underline{A}$ and $\underline{\mathrm{H}}\,\underline{\mathrm{B}}(\underline{A})$ are identical, because $a \in A_{w,s}$ iff $a \in \mathrm{B}(A)_{w,(s)}$ iff $a \in \mathrm{H}\,\mathrm{B}(A)_{w,s}$. $\square$

Proposition 21. *Let* $\coprod_{1\times\between_S}\colon \mathbf{Set}^{S^\star\times S} \longrightarrow \mathbf{Set}^{S^\star\times S^\star}$ *be the functor determined by the mapping* $1\times\between_S$ *from* $S^\star \times S$ *into* $S^\star \times S^\star$ *that to a pair* (w,s) *assigns* $(w,(s))$*. Then for the diagram*



we have that $\underline{\mathrm{Fr}}_{\mathrm{B}} \circ \coprod_{1\times\between_S} \cong \underline{\mathrm{B}} \circ \underline{\mathrm{Fr}}_{\mathrm{H}}$ *and* $\Delta_{1\times\between_S} \circ \mathrm{G_B} = \mathrm{G_H} \circ \underline{\mathrm{H}}$

Corollary 4. *Let Σ be an S-sorted signature. Then the Bénabou algebra $\underline{\mathrm{Pol}}_{\mathrm{B}}(\Sigma)$ is isomorphic to $\underline{\mathrm{Fr}}_{\mathrm{B}}(\coprod_{1\times\between_S}\Sigma)$.*

Proof. Because $\underline{\mathrm{Pol}}_{\mathrm{B}}(\Sigma) = \underline{\mathrm{B}}(\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma))$. □

If we agree that $\mathrm{Eq}_{\mathrm{B}}(\Sigma)$ denotes $\mathrm{Pol}_{\mathrm{B}}(\Sigma)^2$, then the congruence generated in $\underline{\mathrm{Pol}}_{\mathrm{B}}(\Sigma)$ by a subfamily $\mathcal{E}$ of $\mathrm{Eq}_{\mathrm{B}}(\Sigma)$ can be characterized as follows.

Proposition 22. *Let $\mathcal{E}$ be a part of $\mathrm{Eq}_{\mathrm{B}}(\Sigma)$. Then $\mathrm{Cg}_{\underline{\mathrm{Pol}}_{\mathrm{B}}(\Sigma)}(\mathcal{E})$ is the smallest part $\overline{\mathcal{E}}$ of $\mathrm{Pol}_{\mathrm{B}}(\Sigma)$ that contains $\mathcal{E}$ and is such that, for every $u, w, x \in S^\star$ satisfies the following conditions:*

1. Reflexivity. *For every $\mathcal{P} \in \mathrm{Pol}_{\mathrm{B}}(\Sigma)_{w,u}$, $(\mathcal{P},\mathcal{P}) \in \overline{\mathcal{E}}_{w,u}$.*
2. Simmetry. *For every $\mathcal{P}$, $\mathcal{Q} \in \mathrm{Pol}_{\mathrm{B}}(\Sigma)_{w,u}$, if $(\mathcal{P},\mathcal{Q}) \in \overline{\mathcal{E}}_{w,u}$, then $(\mathcal{Q},\mathcal{P}) \in \overline{\mathcal{E}}_{w,u}$.*
3. Transitivity. *For every $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R} \in \mathrm{Pol}_{\mathrm{B}}(\Sigma)_{w,u}$, if $(\mathcal{P},\mathcal{Q})$, $(\mathcal{Q},\mathcal{R}) \in \overline{\mathcal{E}}_{w,u}$, then $(\mathcal{P},\mathcal{R}) \in \overline{\mathcal{E}}_{w,s}$.*
4. Product compatibility. *For every $\mathcal{P}$, $\mathcal{Q} \in \mathrm{Pol}_{\mathrm{B}}(\Sigma)_{u,w}$, if, for every $i \in |w|$, $(P_i, Q_i) \in \overline{\mathcal{E}}_{u,(w_i)}$, then $(\langle P_0,\ldots,P_{|w|-1}\rangle, \langle Q_0,\ldots,Q_{|w|-1}\rangle) \in \overline{\mathcal{E}}_{u,w}$*
5. Substitutivity. *For every $\mathcal{P}$, $\mathcal{Q} \in \mathrm{Pol}_{\mathrm{B}}(\Sigma)_{u,x}$ and $\mathcal{M},\mathcal{N} \in \mathrm{Pol}_{\mathrm{B}}(\Sigma)_{x,w}$, if $(\mathcal{P},\mathcal{Q}) \in \overline{\mathcal{E}}_{u,x}$ and $(\mathcal{M},\mathcal{N}) \in \overline{\mathcal{E}}_{x,w}$, then $(\mathcal{M}\circ\mathcal{P}, \mathcal{N}\circ\mathcal{Q}) = (\mathcal{P}^\sharp\circ\mathcal{M}, \mathcal{Q}^\sharp\circ\mathcal{N}) \in \overline{\mathcal{E}}_{u,w}$.*

Now we define two pairs of order preserving mappings, in opposite directions, between the ordered sets $\underline{\mathrm{Sub}}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))$ and $\underline{\mathrm{Sub}}(\mathrm{Eq}_{\mathrm{B}}(\Sigma))$ that will allow us to assert that the category $\underline{\mathrm{Sub}}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))$ is a retract of $\underline{\mathrm{Sub}}(\mathrm{Eq}_{\mathrm{B}}(\Sigma))$ in the category **Adj** of categories and adjunctions.

Proposition 23. *Let Σ be an S-sorted signature. Then the mappings H, D from $\mathrm{Sub}(\mathrm{Eq}_{\mathrm{B}}(\Sigma))$ into $\mathrm{Sub}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))$ defined as*

$$H(\mathcal{E}) = (\{(P,Q) \in \mathrm{Eq}_{\mathrm{H}}(\Sigma)_{w,s} \mid (P,Q) \in \mathcal{E}_{w,(s)}\})_{(w,s)\in S^\star\times S}$$

$$D(\mathcal{E}) = \left(\left\{(P,Q) \in \mathrm{Eq}_{\mathrm{H}}(\Sigma)_{w,s} \;\middle|\; \begin{array}{c}\exists(\mathcal{R},\mathcal{S}) \in \mathcal{E}_{w,u},\ \exists i \in u^{-1}[s],\\ (P,Q) = (R_i, S_i)\end{array}\right\}\right)_{(w,s)\in S^\star\times S}$$

and the mappings I, B from $\mathrm{Sub}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))$ into $\mathrm{Sub}(\mathrm{Eq}_{\mathrm{B}}(\Sigma))$ defined as

$$I(\mathcal{E}') = (\{\, (P,Q) \in \mathrm{Eq}_{\mathrm{B}}(\Sigma)_{w,u} \mid \exists s \in S\,(u = (s) \text{ and } (P,Q) \in \mathcal{E}'_{w,s})\,\})_{(w,u)\in S^\star\times S^\star}$$

$$B(\mathcal{E}') = (\{\, (\mathcal{P},\mathcal{Q}) \in \mathrm{Eq}_{\mathrm{B}}(\Sigma)_{w,u} \mid \forall i \in |u|\,((P_i,Q_i) \in \mathcal{E}'_{w,u_i})\,\})_{(w,u)\in S^\star\times S^\star}$$

are order preserving. Moreover, $H\circ I = D\circ I = H\circ B = D\circ B = \mathrm{id}_{\mathrm{Sub}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))}$ and, for every $\mathcal{E} \subseteq \mathrm{Eq}_{\mathrm{H}}(\Sigma)$ and $\mathcal{E}' \subseteq \mathrm{Eq}_{\mathrm{B}}(\Sigma)$, we have that $D(\mathcal{E}) \subseteq \mathcal{E}'$ iff $\mathcal{E} \subseteq B(\mathcal{E}')$ and $I(\mathcal{E}') \subseteq \mathcal{E}$ iff $\mathcal{E}' \subseteq H(\mathcal{E})$, hence $D \dashv B$ and $I \dashv H$. Finally, because the composite adjunction $D\circ I \dashv H\circ B$ is the identity adjunction, we conclude that $\underline{\mathrm{Sub}}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))$ is a retract of $\underline{\mathrm{Sub}}(\mathrm{Eq}_{\mathrm{B}}(\Sigma))$ in the category **Adj** *of categories and adjunctions.*

Proposition 24. *Let Σ be an S-sorted signature. Then the lattices $\underline{\mathrm{Cgr}}(\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma))$ and $\underline{\mathrm{Cgr}}(\underline{\mathrm{Pol}}_{\mathrm{B}}(\Sigma))$ are isomorphic.*

Proof. If $\mathcal{E} \in \mathrm{Cgr}(\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma))$ then $\mathrm{Cg}_{\underline{\mathrm{Pol}}_{\mathrm{B}}(\Sigma)}(B(\mathcal{E})) = B(\mathrm{Cg}_{\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)}(\mathcal{E})) \subseteq B(\mathcal{E})$ and $B(\mathcal{E}) \in \mathrm{Cgr}(\underline{\mathrm{Pol}}_{\mathrm{B}}(\Sigma))$.

Reciprocally, if $\mathcal{E} \in \mathrm{Cgr}(\underline{\mathrm{Pol}}_{\mathrm{B}}(\Sigma))$, then $\mathrm{Cg}_{\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma)}(H(\mathcal{E})) \subseteq H(\mathrm{Cg}_{\underline{\mathrm{Pol}}_{\mathrm{B}}(\Sigma)}(\mathcal{E})) \subseteq H(\mathcal{E})$ and $H(\mathcal{E}) \in \mathrm{Cgr}(\underline{\mathrm{Pol}}_{\mathrm{H}}(\Sigma))$. But, because $H\circ B = \mathrm{id}_{\mathrm{Sub}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))}$, we only have to verify that, for every $\mathcal{E} \in \mathrm{Cgr}(\underline{\mathrm{Pol}}_{\mathrm{B}}(\Sigma))$, $B(H(\mathcal{E})) = \mathcal{E}$. If $(\mathcal{P},\mathcal{Q}) \in B(H(\mathcal{E}))_{u,w}$, then, for every $i \in |w|$, $(P_i,Q_i) \in H(\mathcal{E})_{u,w_i}$, hence $(P_i,Q_i) \in \mathcal{E}_{u,(w_i)}$ and $(\mathcal{P},\mathcal{Q}) \in$

$\mathcal{E}_{u,w}$. If $(\mathcal{P},\mathcal{Q}) \in \mathcal{E}_{u,w}$, then, for every $i \in |w|$, $(P_i, Q_i) \in \mathcal{E}_{u,(w_i)}$, hence $(P_i, Q_i) \in H(\mathcal{E})_{u,w_i}$ and $(\mathcal{P},\mathcal{Q}) \in B(H(\mathcal{E}))_{u,w}$. □

Corollary 5 (Completeness Theorem)**.** *Let Σ be an S-sorted signature. Then the algebraic lattice $\underline{\mathrm{Cgr}}(\underline{\mathrm{Pol}}_{\mathrm{B}}(\Sigma))$ is isomorphic to the algebraic lattice of fixed points of* Cn_Σ.

3. Polynomials and equations for monads.

In this section we define, for a monad in a category, the concepts of polynomial, equation and the relation of validation of an equation in an algebra for the monad. From this, as in the classical case, we also obtain a contravariant Galois connection between the ordered class of classes of $\mathbb{T}$-algebras and the ordered set of families of $\mathbb{T}$-equations.

Definition 13. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad in a category $\mathbf{C}$ and X, Y objects in $\mathbf{C}$.

(1) A $\mathbb{T}$-*polynomial of type* (X,Y) is a morphism $P\colon Y \longrightarrow T(X)$ in $\mathbf{C}$. We identify the $\mathbb{T}$-polynomials with the morphisms in $\mathbf{Kl}(\mathbb{T})^{\mathrm{op}}$, the dual of the Kleisli category of $\mathbb{T}$, hence $P\colon X \longrightarrow Y$ in $\mathbf{Kl}(\mathbb{T})^{\mathrm{op}}$ is $P\colon Y \longrightarrow X$ in $\mathbf{Kl}(\mathbb{T})$ or, what amounts to the same, $P\colon Y \longrightarrow T(X)$ in $\mathbf{C}$.
(2) A $\mathbb{T}$-*equation of type* (X,Y) is a pair (P,Q) of $\mathbb{T}$-polynomials of type (X,Y). We identify the $\mathbb{T}$-equations with the parallel pairs of morphisms in $\mathbf{Kl}(\mathbb{T})^{\mathrm{op}}$.

We agree that $\mathbf{Pol}(\mathbb{T})$ denotes the category $\mathbf{Kl}(\mathbb{T})^{\mathrm{op}}$ and call it the category of $\mathbb{T}$-polynomials. On the other hand, $\mathrm{Eq}(\mathbb{T})$ is $(\mathrm{Hom}_{\mathbf{Pol}(\mathbb{T})}(X,Y)^2)_{(X,Y)\in\mathbf{C}^2}$, the $\mathbf{C}^2$-sorted set of $\mathbb{T}$-equations. Moreover, we call the $\mathbf{C}^2$-sorted subsets of $\mathrm{Eq}(\mathbb{T})$, that are the relations on the category $\mathbf{Pol}(\mathbb{T})$, families of $\mathbb{T}$-equations.

To avoid misunderstandings we denote by $\diamond$ the composition in $\mathbf{Kl}(\mathbb{T})$ and $\mathbf{Pol}(\mathbb{T})$, preserving the standard notation for the composition in the category $\mathbf{C}$. Therefore, if $Q\colon Z \longrightarrow T(Y)$ and $P\colon Y \longrightarrow T(X)$ are morphism in $\mathbf{Kl}(\mathbb{T})$, then $P \diamond Q = \mu_X \circ T(P) \circ Q$.

Now we define for a monad in a category, on the one hand, the realization of the polynomials relative to the monad in the algebras for the monad and, on the other, the concept of validation of an equation for the monad in an algebra for the monad.

Definition 14. Let $\mathbb{T}$ be a monad in $\mathbf{C}$ and (A,α) a $\mathbb{T}$-algebra. Then every $\mathbb{T}$-polynomial $P\colon X \longrightarrow Y$ induces a mapping $P^{(A,\alpha)}\colon \mathrm{Hom}_{\mathbf{C}}(X,A) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(Y,A)$, the *realization* of P in (A,α), that to a morphism $f\colon X \longrightarrow A$ assigns the morphism $\alpha \circ T(f) \circ P\colon Y \longrightarrow A$.

From now on, we agree that to say that a diagram of the form

$$a \xrightarrow{\ f\ } b \underset{h}{\overset{g}{\underset{+}{\rightrightarrows}}} c \xrightarrow{\ k\ } d$$

commutes, means that $k \circ g \circ f = k \circ h \circ f$. We extend this convention to similar diagrams.

Definition 15. Let (A,α) be a $\mathbb{T}$-algebra and (P,Q) a $\mathbb{T}$-equation of type (X,Y). We say that (P,Q) is *valid* in (A,α), denoted by $(A,\alpha) \models^{\mathbb{T}}_{X,Y} (P,Q)$, if for every

$f\colon X \longrightarrow A$, $\alpha \circ \mathrm{T}(\alpha) \circ P = \alpha \circ \mathrm{T}(\alpha) \circ Q$, i.e., if the following diagram commutes

$$Y \underset{Q}{\overset{P}{\underset{\longrightarrow}{\overset{\longrightarrow}{+}}}} T(X) \xrightarrow{T(f)} T(A) \xrightarrow{\alpha} A$$

or, equivalently, if $P^{(A,\alpha)} = Q^{(A,\alpha)}$. If $\mathcal{K} \subseteq \mathbf{EM}(\mathbb{T})$, where $\mathbf{EM}(\mathbb{T})$ is the Eilenberg-Moore category of $\mathbb{T}$, then we agree that $\mathcal{K} \models^{\mathbb{T}}_{X,Y} (P,Q)$ means that, for every $(A,\alpha) \in \mathcal{K}$, $(A,\alpha) \models^{\mathbb{T}}_{X,Y} (P,Q)$.

As for general algebra, from the concept of validation we also obtain a contravariant Galois connection.

Definition 16. Let $\mathbb{T}$ be a monad in $\mathbf{C}$.

(1) If $\mathcal{K} \subseteq \mathbf{EM}(\mathbb{T})$, then the $\mathbb{T}$-*equational theory determined by* $\mathcal{K}$, $\mathrm{Th}_{\mathbb{T}}(\mathcal{K})$, is

$$\mathrm{Th}_{\mathbb{T}}(\mathcal{K}) = \left(\left\{(P,Q) \in \mathrm{Eq}(\mathbb{T})_{X,Y} \mid \forall (A,\alpha) \in \mathcal{K}\,((A,\alpha) \models^{\mathbb{T}}_{X,Y} (P,Q))\right\}\right)_{(X,Y)\in \mathbf{C}^2}$$

(2) If $\mathcal{E} \subseteq \mathrm{Eq}(\mathbb{T})$, then the $\mathbb{T}$-*equational class determined by* $\mathcal{E}$, $\mathrm{Mod}_{\mathbb{T}}(\mathcal{E})$, has as elements the $\mathbb{T}$-algebras (A,α) that validate each equation of $\mathcal{E}$, i.e.,

$$\mathrm{Mod}_{\mathbb{T}}(\mathcal{E}) = \left\{ (A,\alpha) \in \mathbf{EM}(\mathbb{T}) \;\middle|\; \begin{array}{c} \forall X,Y \in \mathbf{C},\, \forall (P,Q) \in \mathcal{E}_{X,Y}, \\ (A,\alpha) \models^{\mathbb{T}}_{X,Y} (P,Q) \end{array} \right\}$$

Proposition 25. *Let $\mathbb{T}$ be a monad in $\mathbf{C}$, $\mathcal{E}$, $\mathcal{E}'$ two families of $\mathbb{T}$-equations and $\mathcal{K}$, $\mathcal{K}'$ two classes of $\mathbb{T}$-algebras. Then the following holds:*

(1) *If $\mathcal{E} \subseteq \mathcal{E}'$, then $\mathrm{Mod}_{\mathbb{T}}(\mathcal{E}') \subseteq \mathrm{Mod}_{\mathbb{T}}(\mathcal{E})$.*
(2) *If $\mathcal{K} \subseteq \mathcal{K}'$, then $\mathrm{Th}_{\mathbb{T}}(\mathcal{K}') \subseteq \mathrm{Th}_{\mathbb{T}}(\mathcal{K})$.*
(3) *$\mathcal{E} \subseteq \mathrm{Th}_{\mathbb{T}}(\mathrm{Mod}_{\mathbb{T}}(\mathcal{E}))$ and $\mathcal{K} \subseteq \mathrm{Mod}_{\mathbb{T}}(\mathrm{Th}_{\mathbb{T}}(\mathcal{K}))$.*

Therefore the pair of mappings $\mathrm{Th}_{\mathbb{T}}$ and $\mathrm{Mod}_{\mathbb{T}}$ is a contravariant Galois connection.

The categories associated to the lattices of classes of $\mathbb{T}$-algebras and families of $\mathbb{T}$-equations are related by the adjunction $\mathrm{Mod}_{\mathbb{T}} \dashv \mathrm{Th}_{\mathbb{T}}$, i.e., for every class $\mathcal{K}$ of $\mathbb{T}$-algebras and every family $\mathcal{E}$ of $\mathbb{T}$-equations, we have that $\mathcal{K} \subseteq \mathrm{Mod}_{\mathbb{T}}(\mathcal{E})$ iff $\mathcal{E} \subseteq \mathrm{Th}_{\mathbb{T}}(\mathcal{K})$, because of the contravariance.

Definition 17. Let $\mathbb{T}$ be a monad in $\mathbf{C}$. We denote by $\mathrm{Cn}_{\mathbb{T}}$ the closure operator $\mathrm{Th}_{\mathbb{T}} \circ \mathrm{Mod}_{\mathbb{T}}$ on $\mathrm{Eq}(\mathbb{T})$ and we call the $\mathrm{Cn}_{\mathbb{T}}$-closed sets $\mathbb{T}$-*equational theories*. If $\mathcal{E}$ is a family of $\mathbb{T}$-equations and (P,Q) a $\mathbb{T}$-equation of type (X,Y), then we say that (P,Q) is a *semantical consequence* of $\mathcal{E}$ if $\mathrm{Mod}_{\mathbb{T}}(\mathcal{E}) \subseteq \mathrm{Mod}_{\mathbb{T}}(P,Q)$, i.e., if $(P,Q) \in \mathrm{Th}_{\mathbb{T}}(\mathrm{Mod}_{\mathbb{T}}(\mathcal{E}))_{X,Y}$.

4. The completeness theorem for monads in categories of sorted sets.

In this last section once defined, for a congruence on a category, the concept of $\varprojlim$-compatible congruence, and his particular case that of Π-compatible congruence, we prove the completeness theorem for a monad in a category of sorted sets, in the version that says that the lattice of Π-compatible congruences on the category of polynomials for a monad in a category of sorted sets is identical to the lattice of equational theories for the monad. But before that, because we need the quotient algebras to prove the completeness theorem, we define and characterize, for a monad in a category of sorted sets, the concept of congruence on an algebra in the Eilenberg-Moore category for the monad.

Definition 18. Let S be a set of sorts, $\mathbb{T}$ a monad in $\mathbf{Set}^S$, (A,α) a $\mathbb{T}$-algebra and Φ an equivalence on A. We say that Φ is a *congruence* on (A,α) if there is a $\xi\colon T(\Phi) \longrightarrow \Phi$ such that

(1) (Φ, ξ) is a $\mathbb{T}$-algebra.
(2) The restrictions p^0 and p^1 to Φ of the canonical projections pr^0 and pr^1 from $A \times A$ to A are $\mathbb{T}$-morphisms from (Φ, ξ) to (A, α).

If Φ is a congruence on (A, α), then we denote by $(A/\Phi, \alpha/\Phi)$ the $\mathbb{T}$-*quotient* algebra.

Proposition 26. *Let S be a set of sorts, $\mathbb{T}$ a monad in $\mathbf{Set}^S$, (A, α) a $\mathbb{T}$-algebra and Φ an equivalence on A. Then Φ is a congruence on (A, α) iff the following diagram commutes:*

$$T(\Phi) \underset{T(\mathrm{p}^1)}{\overset{T(\mathrm{p}^0)}{\rightrightarrows}} T(A) \xrightarrow{\ \alpha\ } A \xrightarrow{\ \mathrm{pr}^\Phi\ } A/\Phi$$

and for the unique $\xi \colon T(\Phi) \longrightarrow \Phi$ such that $\mathrm{p}^0 \circ \xi = \alpha \circ T(\mathrm{p}^0)$ and $\mathrm{p}^1 \circ \xi = \alpha \circ T(\mathrm{p}^1)$, (Φ, ξ) is a $\mathbb{T}$-algebra.

Proposition 27. *Let S be a set of sorts, $\mathbb{T}$ a monad in $\mathbf{Set}^S$, (A, α) a $\mathbb{T}$-algebra and Φ an equivalence on A. Then are equivalent:*

(1) *The following diagram commutes*

$$T(\Phi) \underset{T(\mathrm{p}^1)}{\overset{T(\mathrm{p}^0)}{\rightrightarrows}} T(A) \xrightarrow{\ \alpha\ } A \xrightarrow{\ \mathrm{pr}^\Phi\ } A/\Phi$$

(2) *For every $a, b \colon Y \longrightarrow A$ if $\mathrm{pr}^\Phi \circ a = \mathrm{pr}^\Phi \circ b$, then the following diagram commutes*

$$T(Y) \underset{T(b)}{\overset{T(a)}{\rightrightarrows}} T(A) \xrightarrow{\ \alpha\ } A \xrightarrow{\ \mathrm{pr}^\Phi\ } A/\Phi$$

Proof. Let us suppose that $\mathrm{pr}^\Phi \circ \alpha \circ T(\mathrm{p}^0) = \mathrm{pr}^\Phi \circ \alpha \circ T(\mathrm{p}^1)$ and let $a, b \colon Y \longrightarrow A$ be such that $\mathrm{pr}^\Phi \circ a = \mathrm{pr}^\Phi \circ b$. Then, because (Φ, p^0, p^1) is the kernel pair of $\mathrm{pr}^\Phi \colon A \longrightarrow A/\Phi$, there is a unique $f \colon Y \longrightarrow \Phi$ such that $p^0 \circ f = a$ and $p^1 \circ f = b$. From this follows that the following diagram

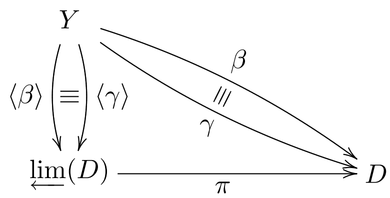
$$T(Y) \xrightarrow{\ T(f)\ } T(\Phi) \underset{T(p^1)}{\overset{T(p^0)}{\rightrightarrows}} T(A) \xrightarrow{\ \alpha\ } A \xrightarrow{\ \mathrm{pr}^\Phi\ } A/\Phi \qquad (T(a), T(b) \colon T(Y) \longrightarrow T(A))$$

commutes. The reciprocal is obvious. □

Definition 19. Let $\mathbf{C}$ be a category, $\mathcal{E}$ a congruence on $\mathbf{C}$, $\mathbf{I}$ a category and $D \colon \mathbf{I} \longrightarrow \mathbf{C}$ a diagram of type $\mathbf{I}$ in $\mathbf{C}$, i.e., a functor from $\mathbf{I}$ in $\mathbf{C}$.

(1) Given two projective cones (Y, β), (Y, γ) from an object Y of $\mathbf{C}$ to D we say that β and γ are $\mathcal{E}$-*congruent*, denoted by $\beta \equiv_{\mathcal{E}} \gamma$, if, for every $i \in \mathbf{I}$, we have that $(\beta_i, \gamma_i) \in \mathcal{E}_{Y, D_i}$.
(2) We say that $\mathcal{E}$ is a $\varprojlim(D)$-*compatible* congruence on $\mathbf{C}$ if, for every projective limit $(\varprojlim(D), \pi)$ of D and every pair of projective cones (Y, β), (Y, γ) from an object Y of $\mathbf{C}$ to D, if $\beta \equiv_{\mathcal{E}} \gamma$, then $\langle \beta \rangle, \langle \gamma \rangle : Y \longrightarrow \varprojlim(D)$, the

unique morphisms such that, for every $i \in \mathbf{I}$, $\pi_i \circ \langle\beta\rangle = \beta_i$ and $\pi_i \circ \langle\gamma\rangle = \gamma_i$, are $\mathcal{E}$-congruents, i.e., $(\langle\beta\rangle, \langle\gamma\rangle) \in \mathcal{E}_{Y,\varprojlim(D)}$.



We say that $\mathcal{E}$ is a $\Pi(D)$-*compatible* congruence on $\mathbf{C}$ if $\mathcal{E}$ is a $\varprojlim(D)$-compatible congruence on $\mathbf{C}$ when $\mathbf{I}$ is discrete.

(3) We say that $\mathcal{E}$ is a $\varprojlim$-*compatible* congruence if, for every $D\colon \mathbf{I} \longrightarrow \mathbf{C}$, $\mathcal{E}$ is a $\varprojlim(D)$-*compatible* congruence on $\mathbf{C}$. If, for every discrete diagram D, $\mathcal{E}$ is a $\Pi(D)$-compatible congruence, then we say that $\mathcal{E}$ is a Π-*compatible* congruence on $\mathbf{C}$.

We remark that the concepts in the above Definition can be dualized. Moreover, the behaviour relative to the morphisms of the $\varprojlim$-compatible congruences is like that of the algebraical congruences relative to the homomorphisms, as stated in the following Proposition.

Proposition 28. *Let $\mathbf{C}$ be a category, $D, D'\colon \mathbf{I} \longrightarrow \mathbf{C}$ two diagrams of type $\mathbf{I}$ in $\mathbf{C}$, $\mathcal{E}$ a $\varprojlim(D)$ and $\varprojlim(D')$-compatible congruence on $\mathbf{C}$ and $\sigma, \tau\colon D \Longrightarrow D'$ two natural transformations from D to D'. Then, for every projective limits $(\varprojlim(D), \pi)$ of D and $(\varprojlim(D'), \pi')$ of D', the unique morphisms $\langle\sigma \circ \pi\rangle, \langle\tau \circ \pi\rangle\colon \varprojlim(D) \longrightarrow \varprojlim(D')$ such that, for every $i \in \mathbf{I}$, $\pi'_i \circ \langle\sigma \circ \pi\rangle = \sigma_i \circ \pi_i$ and $\pi'_i \circ \langle\tau \circ \pi\rangle = \tau_i \circ \pi_i$, are $\mathcal{E}$-congruents.*

Corollary 6. *Let $\mathbf{C}$ be a category, $(A_i)_{i\in I}$, $(B_i)_{i\in I}$ two families of objects in $\mathbf{C}$, $\mathcal{E}$ a Π-compatible congruence on $\mathbf{C}$, and $(f_i)_{i\in I}$, $(g_i)_{i\in I}$ two families of morphisms from $(A_i)_{i\in I}$ to $(B_i)_{i\in I}$. If, for every $i \in I$, f_i and g_i are $\mathcal{E}$-congruent, then $\prod_{i\in I} f_i$ and $\prod_{i\in I} g_i$ are $\mathcal{E}$-congruent.*

Proposition 29. *Let $\mathbf{C}$ be a category with products. Then the ordered set of Π-compatible congruences on $\mathbf{C}$, $\underline{\mathrm{Cgr}}^{\Pi}(\mathbf{C}) = (\mathrm{Cgr}^{\Pi}(\mathbf{C}), \subseteq)$, is a complete lattice.*

Definition 20. Let $\mathbf{C}$ be a category with products. We denote by $\mathrm{Cg}^{\Pi}_{\mathbf{C}}$ the closure operator on the set of relations on $\mathbf{C}$ that to a relation $\mathcal{E}$ on $\mathbf{C}$ assigns the smallest Π-compatible congruence on $\mathbf{C}$ that contains $\mathcal{E}$.

Now we prove that the Kleisli category of a monad $\mathbb{T}$ in a category $\mathbf{C}$ has coproducts if $\mathbf{C}$ has coproducts. From this follows that, for every set of sorts S and monad $\mathbb{T}$ in $\mathbf{Set}^S$, the category $\mathbf{Pol}(\mathbb{T})$ has products, because it is the dual of $\mathbf{Kl}(\mathbb{T})$, therefore on the category $\mathbf{Pol}(\mathbb{T})$ we have the corresponding closure operator $\mathrm{Cg}^{\Pi}_{\mathbf{Pol}(\mathbb{T})}$ that we will use to prove the Completeness Theorem for monads in categories of sorted sets.

Proposition 30. *Let $\mathbb{T}$ be a monad in $\mathbf{C}$. If $\mathbf{C}$ has coproducts, then $\mathbf{Kl}(\mathbb{T})$ has coproducts.*

Proof. Let $(X_i)_{i\in I}$ be a family of objects in $\mathbf{Kl}(\mathbb{T})$. Then $\coprod_{i\in I} X_i$, together with the family of morphisms $(\eta_{\coprod_{i\in I} X_i} \circ \mathrm{in}_i)_{i\in I}$, is a coproduct in $\mathbf{Kl}(\mathbb{T})$ of $(X_i)_{i\in I}$.

Let $(f_i\colon X_i \longrightarrow Y)_{i\in I}$ be a family of morphisms in $\mathbf{Kl}(\mathbb{T})$. Then we have, in $\mathbf{C}$, the commutative diagram

$$\begin{array}{ccccc}
X_i & \xrightarrow{\mathrm{in}_i} & \coprod_{i\in I} X_i & \xrightarrow{\eta_{\coprod_{i\in I} X_i}} & T(\coprod_{i\in I} X_i) \\
& \searrow^{f_i} & \big\downarrow [f_i]_{i\in I} & & \big\downarrow T([f_i]_{i\in I}) \\
& & T(Y) & \xleftarrow[\mu_Y]{} & T(T(Y))
\end{array}$$

and $[f_i]_{i\in I}\colon \coprod_{i\in I} X_i \longrightarrow Y$ in $\mathbf{Kl}(\mathbb{T})$ satisfies the universal property. $\square$

Corollary 7. *Let $\mathbb{T}$ be a monad in $\mathbf{C}$. If $\mathbf{C}$ has coproducts, then $\mathbf{Pol}(\mathbb{T})$ has products. Therefore, for every set of sorts S and every monad $\mathbb{T}$ in $\mathbf{Set}^S$, $\mathbf{Pol}(\mathbb{T})$ has products.*

Next we prove that the Π-compatible congruences on the category of polynomials for a monad in a category of sorted sets, are determined by the pairs of morphisms in the congruence with codomains deltas of Kronecker. Moreover, from now on, for a monad $\mathbb{T} = (T, \eta, \mu)$ in $\mathbf{Set}^S$ we denote by η^X and μ^X the values of η and μ, respectively, in the S-sorted set X.

Proposition 31. *Let $\mathbb{T}$ be a monad in $\mathbf{Set}^S$ and $\mathcal{E}$ a Π-compatible congruence on $\mathbf{Pol}(\mathbb{T})$. Then $(P, Q) \in \mathcal{E}_{X,Y}$ iff, for every $s \in S$ and $(y)\colon \delta^s \longrightarrow Y$ in $\mathbf{Set}^S$, we have that $(P \circ (y), Q \circ (y)) \in \mathcal{E}_{X,\delta^s}$.*

Proof. Before we proceed to the proof, we remark that, for every $(y)\colon \delta^s \longrightarrow Y$, $\eta^Y \circ (y)$ is a morphism in $\mathbf{Pol}(\mathbb{T})$ from Y to δ^s. Moreover, if $R\colon X \longrightarrow Y$ is another morphism in $\mathbf{Pol}(\mathbb{T})$, then $(\eta^Y \circ (y)) \diamond R = R \circ (y)$.

If $(P, Q) \in \mathcal{E}_{X,Y}$, then, for every $s \in S$ and $(y)\colon \delta^s \longrightarrow Y$, because $\mathcal{E}$ is a congruence, we have that $(P \circ (y), Q \circ (y)) \in \mathcal{E}_{X,\delta^s}$.

Reciprocally, if, for every $s \in S$ and $(y)\colon \delta^s \longrightarrow Y$, $(P \circ (y), Q \circ (y)) \in \mathcal{E}_{X,\delta^s}$, then, because $(Y, (\eta^Y \circ (y))_{s\in S, y\in Y_s})$ is a product in $\mathbf{Pol}(\mathbb{T})$ of $(\delta^s)_{s\in S, y\in Y_s}$ and $\mathcal{E}$ is Π-compatible, it follows that the pair $(\langle P \circ (y)\rangle_{s\in S, y\in Y_s}, \langle Q \circ (y)\rangle_{s\in S, y\in Y_s}) \in \mathcal{E}_{X,Y}$. But, by the universal property of the product, $(\langle P\circ(y)\rangle_{s\in S, y\in Y_s}, \langle Q\circ(y)\rangle_{s\in S, y\in Y_s}) = (P, Q)$, hence $(P, Q) \in \mathcal{E}_{X,Y}$. $\square$

Next we prove the Soundness Theorem, i.e., that for every subclass $\mathcal{K}$ of the Eilenberg-Moore category for the monad $\mathbb{T}$, the $\mathbb{T}$-equational theory $\mathrm{Th}_{\mathbb{T}}(\mathcal{K})$ is a Π-compatible congruence on $\mathbf{Pol}(\mathbb{T})$.

Theorem 1 (Soundness Theorem)**.** *Let S be a set and $\mathbb{T}$ a monad in $\mathbf{Set}^S$. Then every $\mathbb{T}$-equational theory is a Π-compatible congruence on $\mathbf{Pol}(\mathbb{T})$.*

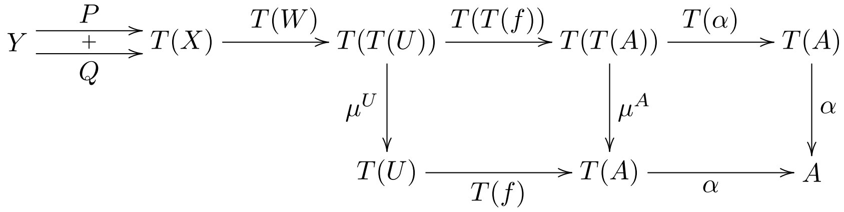
Proof. Let $\mathrm{Th}_{\mathbb{T}}(\mathcal{K})$ be a $\mathbb{T}$-equational theory for some $\mathcal{K} \subseteq \mathbf{EM}(\mathbb{T})$. Then, for every X, $Y \in \mathbf{Set}^S$, $\mathrm{Th}_{\mathbb{T}}(\mathcal{K})_{X,Y}$ is an equivalence on $\mathrm{Hom}_{\mathbf{Pol}(\mathbb{T})}(X, Y)$.

Now we prove that the equivalence $\mathrm{Th}_{\mathbb{T}}(\mathcal{K})$ is compatible with the composition in $\mathbf{Pol}(\mathbb{T})$. Let $(P, Q) \in \mathrm{Th}_{\mathbb{T}}(\mathcal{K})_{X,Y}$ be and $R\colon Y \longrightarrow Z$ a morphism in $\mathbf{Pol}(\mathbb{T})$. Then, for every $\mathbb{T}$-algebra (A, α) and morphism $f\colon X \longrightarrow A$, the following diagram in $\mathbf{Set}^S$ commutes:

$$\begin{array}{ccccccccc}
& & Y & \overset{P}{\underset{Q}{\rightrightarrows}} & T(X) & \xrightarrow{T(f)} & T(A) & \xrightarrow{\alpha} & A \\
& & & & \big\uparrow \mu^X & & \big\uparrow \mu^A & & \big\uparrow \alpha \\
Z & \xrightarrow[R]{} & T(Y) & \overset{T(P)}{\underset{T(Q)}{\rightrightarrows}} & T(T(X)) & \xrightarrow[T(T(f))]{} & T(T(A)) & \xrightarrow[T(\alpha)]{} & T(A)
\end{array}$$

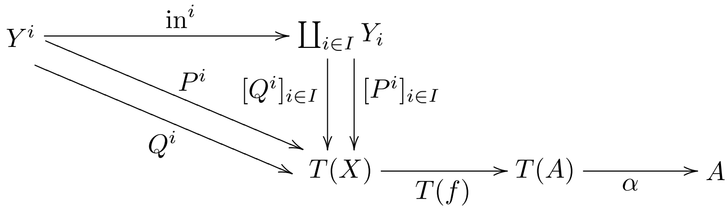
Therefore, $(R \diamond P, R \diamond Q) \in \mathrm{Th}_\Sigma(\mathcal{K})_{X,Z}$.

Let $W\colon U \longrightarrow X$ be a morphism in $\mathbf{Pol}(\mathbb{T})$. Then, for every $\mathbb{T}$-algebra (A, α) and morphism $f\colon U \longrightarrow A$, the following diagram commutes



Therefore $(P \diamond W, Q \diamond W) \in \mathrm{Th}_\Sigma(\mathcal{K})_{U,Y}$.

Lastly, we prove that $\mathrm{Th}_\mathbb{T}(\mathcal{K})$ is Π-compatible. Let $(P^i)_{i\in I}$ and $(Q^i)_{i\in I}$ be two families of morphisms in $\mathbf{Pol}(\mathbb{T})$ such that, for every $i \in I$, $(P^i, Q^i) \in \mathrm{Th}_\mathbb{T}(\mathcal{K})_{X,Y^i}$, (A, α) a $\mathbb{T}$-algebra in $\mathcal{K}$ and $f\colon X \longrightarrow A$ a morphism in $\mathbf{Set}^S$. Then we have the following diagram in $\mathbf{Set}^S$



For every $i \in I$, let f^i be the morphism $\alpha \circ T(f) \circ P^i = \alpha \circ T(f) \circ Q^i$. Then, by the universal property of $\coprod_{i\in I} Y^i$, there exists a unique $[f^i]_{i\in I}\colon \coprod_{i\in I} Y^i \longrightarrow A$ such that, for every $i \in I$, $[f^i]_{i\in I} \circ \mathrm{in}^i = f^i$. Moreover, for every $i \in I$, we have that $\alpha \circ T(f) \circ [P^i]_{i\in I} \circ \mathrm{in}^i = f^i = \alpha \circ T(f) \circ [Q^i]_{i\in I} \circ \mathrm{in}^i$, hence $\alpha \circ T(f) \circ [P^i]_{i\in I} = \alpha \circ T(f) \circ [Q^i]_{i\in I}$, therefore $([P^i]_{i\in I}, [Q^i]_{i\in I}) \in \mathrm{Th}_\mathbb{T}(\mathcal{K})_{X,\coprod_{i\in I} Y^i}$. □

We remark that the Soundness Theorem is equivalent to $\mathrm{Cg}^{\Pi}_{\mathbf{Pol}(\mathbb{T})} \leq \mathrm{Cn}_\mathbb{T}$, because the lattices of closure operators and closure systems on a set are anti-isomorphic.

Now in order to prove the Adequacy Theorem, i.e., that every Π-compatible congruence $\mathcal{E}$ on $\mathbf{Pol}(\mathbb{T})$ is identical to $\mathrm{Th}_\mathbb{T}(\mathcal{K})$ for some class $\mathcal{K}$ of $\mathbb{T}$-algebras, we begin by associating to every family $\mathcal{E}$ of $\mathbb{T}$-equations and S-sorted set X a many-sorted relation $\mathcal{E}_X$ on $T(X)$ in such a way that for $\mathcal{E}$ a Π-compatible congruence on $\mathbf{Pol}(\mathbb{T})$, $\mathcal{E}_X$ will be a congruence on $(T(X), \mu^X)$. Then to obtain the Theorem just mentioned it will be enough to take as $\mathcal{K} = \{(T(X)/\mathcal{E}_X, \mu^X/\mathcal{E}_X) \mid X \in \boldsymbol{\mathcal{U}}^S\}$.

Definition 21. Let $\mathbb{T}$ be a monad in $\mathbf{Set}^S$, $\mathcal{E}$ a family of $\mathbb{T}$-equations and X an S-sorted set. Then we denote by $\mathcal{E}_X$ the many-sorted relation on $T(X)$ defined as follows

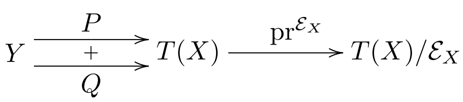
$$\mathcal{E}_X = (\{(p, q) \in T(X)^2_s \mid ((p), (q)) \in \mathcal{E}_{X,\delta^s}\})_{s\in S},$$

where, for $s \in S$ and $p \in T(X)_s$, (p) is the associated mapping from δ^s to $T(X)$.

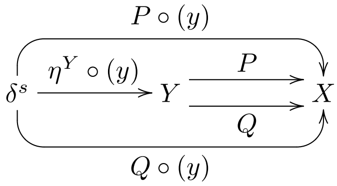
Lemma 2. *Let $\mathbb{T}$ a monad in $\mathbf{Set}^S$, $\mathcal{E}$ a Π-compatible congruence on $\mathbf{Pol}(\mathbb{T})$ and X an S-sorted set. Then, for every S-sorted set Y and every $(P, Q) \in \mathrm{Eq}(\mathbb{T})_{X,Y}$, the following conditions are equivalents:*

(1) $(P, Q) \in \mathcal{E}_{X,Y}$.

(2) *The following diagram commutes*



Proof. Let us suppose that $(P,Q) \in \mathcal{E}_{X,Y}$. Then, for every $s \in S$ and $(y)\colon \delta^s \longrightarrow Y$ in $\mathbf{Set}^S$, in the following diagram in $\mathbf{Kl}(\mathbb{T})$

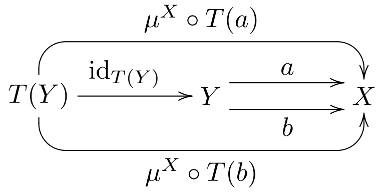


we have that $P \circ (y) = P \diamond (\eta^Y \circ (y))$ and $Q \circ (y) = Q \diamond (\eta^Y \circ (y))$, by the definition of $\diamond$ and because η is natural and $\mu^X \circ \eta^{T(X)}$ is the identity in $T(X)$. Therefore, given that $\mathcal{E}$ is a congruence, $(P \circ (y), Q \circ (y)) \in \mathcal{E}_{X,\delta^s}$, hence $(P_s(y), Q_s(y)) \in \mathcal{E}_{X,s}$. From this and taking into account that the deltas of Kronecker are a set of generators for $\mathbf{Set}^S$, follows that $\mathrm{pr}^{\mathcal{E}_X} \circ P = \mathrm{pr}^{\mathcal{E}_X} \circ Q$.

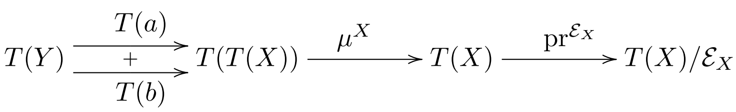
Reciprocally, if $\mathrm{pr}^{\mathcal{E}_X} \circ P = \mathrm{pr}^{\mathcal{E}_X} \circ Q$, then, for every $(y)\colon \delta^s \longrightarrow Y$ and $s \in S$, $\mathrm{pr}^{\mathcal{E}_X} \circ P \circ (y) = \mathrm{pr}^{\mathcal{E}_X} \circ Q \circ (y)$, hence $(P \circ (y), Q \circ (y)) \in \mathcal{E}_{X,\delta^s}$. Therefore, by the Proposición 31, $(P,Q) \in \mathcal{E}_{X,Y}$. □

Proposition 32. *Let $\mathbb{T}$ be a monad in $\mathbf{Set}^S$ and $\mathcal{E}$ a Π-compatible congruence on $\mathbf{Pol}(\mathbb{T})$. Then, for every S-sorted set X, $\mathcal{E}_X$ is a congruence on $(T(X), \mu^X)$.*

Proof. The many-sorted relation $\mathcal{E}_X$ is an equivalence on $T(X)$ because $\mathcal{E}$ is, in particular, an equivalence. Now, by the Proposition 27, instead of proving that $\mathrm{pr}^{\mathcal{E}_X} \circ \mu^X \circ T(\mathrm{p}^0) = \mathrm{pr}^{\mathcal{E}_X} \circ \mu^X \circ T(\mathrm{p}^1)$, we prove that, for every $a, b\colon Y \longrightarrow T(X)$, if $\mathrm{pr}^{\mathcal{E}_X} \circ a = \mathrm{pr}^{\mathcal{E}_X} \circ b$, then $\mathrm{pr}^{\mathcal{E}_X} \circ \mu^X \circ T(a) = \mathrm{pr}^{\mathcal{E}_X} \circ \mu^X \circ T(b)$. Let $a, b\colon Y \longrightarrow T(X)$ be two S-sorted mappings such that $\mathrm{pr}^{\mathcal{E}_X} \circ a = \mathrm{pr}^{\mathcal{E}_X} \circ b$. Then, by the Lemma 2, we have that $(a,b) \in \mathcal{E}_{X,Y}$. But for the following diagram in $\mathbf{Kl}(\mathbb{T})$



we have that $\mu^X \circ T(a) = a \diamond \mathrm{id}_{T(Y)}$ and $\mu^X \circ T(b) = b \diamond \mathrm{id}_{T(Y)}$. Therefore, because $\mathcal{E}$ is a congruence on $\mathbf{Pol}(\mathbb{T})$, $(\mu^X \circ T(a), \mu^X \circ T(b)) \in \mathcal{E}_{X,T(Y)}$. Hence, once more, by the Lemma 2, the following diagram in $\mathbf{Set}^S$

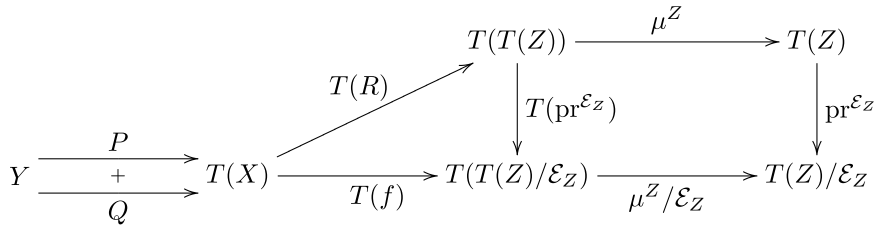


commutes. Moreover, $(\mathcal{E}_X, \xi)$, where ξ is the unique morphism from $T(\mathcal{E}_X)$ to $\mathcal{E}_X$ such that $\mathrm{p}^0 \circ \xi = \mu^X \circ T(\mathrm{p}^0)$ and $\mathrm{p}^1 \circ \xi = \mu^X \circ T(\mathrm{p}^1)$, is a $\mathbb{T}-$subalgebra of $(T(X), \mu^X)^2$. Therefore $\mathcal{E}_X$ is a congruence on $(T(X), \mu^X)$. □

Proposition 33. *Let $\mathbb{T}$ be a monad in $\mathbf{Set}^S$, $\mathcal{E}$ a Π-compatible congruence on $\mathbf{Pol}(\mathbb{T})$, X, Y two S-sorted sets and $(P,Q) \in \mathcal{E}_{X,Y}$. Then, for every S-sorted set Z, $(T(Z)/\mathcal{E}_Z, \mu^Z/\mathcal{E}_Z) \models^{\mathbb{T}}_{X,Y} (P,Q)$.*

Proof. Let $f\colon X \longrightarrow T(Z)/\mathcal{E}_Z$ be a valuation. Then, because $\mathrm{pr}^{\mathcal{E}_Z}$ is a retraction, there exists an $R\colon X \longrightarrow T(Z)$ such that $f = \mathrm{pr}^{\mathcal{E}_Z} \circ R$. Henceforth, $(P \diamond R, Q \diamond R) \in$

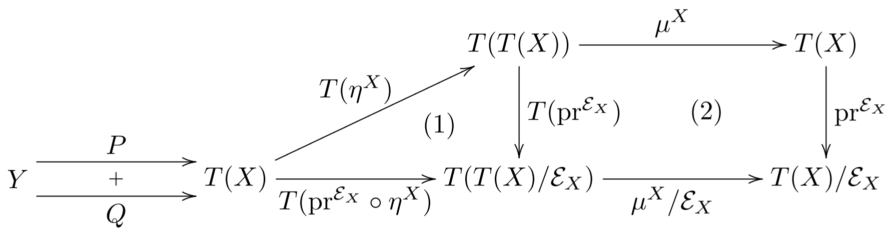
$\mathcal{E}_{Z,Y}$, and by the Lemma 2, $\mathrm{pr}^{\mathcal{E}_Z} \circ \mu^Z \circ T(R) \circ P = \mathrm{pr}^{\mathcal{E}_Z} \circ \mu^Z \circ T(R) \circ Q$. From this follows the commutativity of the following diagram



Therefore $(T(Z)/\mathcal{E}_Z, \mu^Z/\mathcal{E}_Z) \models^{\mathbb{T}}_{X,Y} (P, Q)$. □

Theorem 2 (Adequacy Theorem)**.** *Let S be a set and $\mathbb{T}$ a monad in $\mathbf{Set}^S$. Then every Π-compatible congruence on $\mathbf{Pol}(\mathbb{T})$ is a $\mathbb{T}$-equational theory.*

Proof. Let $\mathcal{E}$ be Π-compatible congruence on $\mathbf{Pol}(\mathbb{T})$. We will prove that $\mathcal{E} = \mathrm{Th}_{\mathbb{T}}(\mathcal{K})$, where $\mathcal{K} = \{(T(X)/\mathcal{E}_X, \mu^X/\mathcal{E}_X) \mid X \in \boldsymbol{\mathcal{U}}^S\}$. Let X, Y be two S-sorted sets and $(P, Q) \in \mathcal{E}_{X,Y}$, then, by Proposition 33, $\mathcal{K} \models^{\mathbb{T}}_{X,Y} (P, Q)$, hence $\mathcal{E} \subseteq \mathrm{Th}_{\mathbb{T}}(\mathcal{K})$. Reciprocally, for $(P, Q) \in \mathrm{Th}_{\mathbb{T}}(\mathcal{K})_{X,Y}$ and $\mathrm{pr}^{\mathcal{E}_X} \circ \eta^X \colon X \longrightarrow T(X)/\mathcal{E}_X$, in the following diagram



the triangle (1), the square (2) and the bottom row commute. Moreover, we have that $\mu^X \circ T(\eta^X) = \mathrm{id}_X$, hence $\mathrm{pr}^{\mathcal{E}_X} \circ P = \mathrm{pr}^{\mathcal{E}_X} \circ Q$. Therefore, by Lemma 2, $(P, Q) \in \mathcal{E}_{X,Y}$. □

We remark that the Adequacy Theorem is equivalent to $\mathrm{Cn}_{\mathbb{T}} \leq \mathrm{Cg}^{\Pi}_{\mathbf{Pol}(\mathbb{T})}$, because the lattices of closure operators and closure systems on a set are anti-isomorphic.

Corollary 8 (Completeness Theorem)**.** *Let S be a set and $\mathbb{T}$ a monad in $\mathbf{Set}^S$. Then, the lattice of Π-compatible congruences on $\mathbf{Pol}(\mathbb{T})$ and the lattice of the $\mathbb{T}$-equational theories are identical or, what is equivalent,* $\mathrm{Cn}_{\mathbb{T}} = \mathrm{Cg}^{\Pi}_{\mathbf{Pol}(\mathbb{T})}$.

As we know, for a set of sorts S and a monad $\mathbb{T}$ in $\mathbf{Set}^S$, the category $\mathbf{Pol}(\mathbb{T})$ of $\mathbb{T}$-polynomials has a set of cogenerators, the deltas of Kronecker, and is a category with products. From this follows that a $\mathbb{T}$-equation $(P, Q) \in \mathrm{Eq}(\mathbb{T})_{X,Y}$ is valid in a $\mathbb{T}$-algebra (A, α) iff every equation in $\mathrm{Eq}(\mathbb{T})_{X,\delta^s}$ obtained from (P, Q) by composition with a morphism $R \colon Y \longrightarrow \delta^s$ in $\mathbf{Pol}(\mathbb{T})$, is valid in (A, α). This fact allows us, without loss of generality, to restrict, for monads in categories of sorted sets, to consider exclusively equations which have as codomain a delta of Kronecker δ^s, for some sort $s \in S$. Moreover, for this type of equation, we have a consequence operator directly definible and equivalent to the operator $\mathrm{Cg}^{\Pi}_{\mathbf{Pol}(\mathbb{T})}$.

Definition 22. Let $\mathbb{T}$ be a monad in $\mathbf{Set}^S$ and $\mathrm{Eq}^{\delta}(\mathbb{T})$ the $\boldsymbol{\mathcal{U}}^S \times S$-sorted family $(\mathrm{Hom}_{\mathbf{Pol}(\mathbb{T})}(X, \delta^s)^2)_{(X,s) \in \boldsymbol{\mathcal{U}}^S \times S}$. Then $\mathrm{Mod}^{\delta}_{\mathbb{T}}$ and $\mathrm{Th}^{\delta}_{\mathbb{T}}$ are the operators defined as follows:

$$\mathrm{Mod}^{\delta}_{\mathbb{T}}\begin{cases}\mathrm{Sub}(\mathrm{Eq}^{\delta}(\mathbb{T})) \longrightarrow \mathrm{Sub}(\mathbf{EM}(\mathbb{T}))\\ \mathcal{D} \mapsto \left\{(A,\alpha)\in\mathbf{EM}(\mathbb{T}) \;\middle|\; \begin{array}{c}\forall (X,s)\in \boldsymbol{\mathcal{U}}^S\times S,\ \forall (P,Q)\in\mathcal{D}_{X,s},\\ (A,\alpha)\models^{\mathbb{T}}_{X,\delta^s}(P,Q)\end{array}\right\}\end{cases}$$

$$\mathrm{Th}^{\delta}_{\mathbb{T}}\begin{cases}\mathrm{Sub}(\mathbf{EM}(\mathbb{T})) \longrightarrow \mathrm{Sub}(\mathrm{Eq}^{\delta}(\mathbb{T}))\\ \mathcal{K} \mapsto \left(\left\{(P,Q)\in\mathrm{Eq}^{\delta}(\mathbb{T})_{X,s} \;\middle|\; \begin{array}{c}\forall (A,\alpha)\in\mathcal{K},\\ (A,\alpha)\models^{\mathbb{T}}_{X,\delta^s}(P,Q)\end{array}\right\}\right)_{(X,s)\in\boldsymbol{\mathcal{U}}^S\times S}\end{cases}$$

The pair of mappings $\mathrm{Th}^{\delta}_{\mathbb{T}}$ and $\mathrm{Mod}^{\delta}_{\mathbb{T}}$ is a contravariant Galois connection.

Proposition 34. *Let $\mathbb{T}$ be a monad in $\mathbf{Set}^S$. Then the operators I, H, D and B, defined as:*

$$I\begin{cases}\mathrm{Sub}(\mathrm{Eq}^{\delta}(\mathbb{T})) \longrightarrow \mathrm{Sub}(\mathrm{Eq}(\mathbb{T}))\\ \mathcal{D} \mapsto \left(\left\{(P,Q)\in\mathrm{Eq}(\mathbb{T})_{X,Y} \;\middle|\; \exists s\in S\,(Y=\delta^s \;\&\; (P,Q)\in\mathcal{D}_{X,s})\right\}\right)_{(X,Y)\in(\boldsymbol{\mathcal{U}}^S)^2}\end{cases}$$

$$H\begin{cases}\mathrm{Sub}(\mathrm{Eq}(\mathbb{T})) \longrightarrow \mathrm{Sub}(\mathrm{Eq}^{\delta}(\mathbb{T}))\\ \mathcal{E} \mapsto \left(\left\{(P,Q)\in\mathrm{Eq}^{\delta}(\mathbb{T})_{X,s} \;\middle|\; (P,Q)\in\mathcal{E}_{X,\delta^s}\right\}\right)_{(X,s)\in\boldsymbol{\mathcal{U}}^S\times S}\end{cases}$$

$$D\begin{cases}\mathrm{Sub}(\mathrm{Eq}(\mathbb{T})) \longrightarrow \mathrm{Sub}(\mathrm{Eq}^{\delta}(\mathbb{T}))\\ \mathcal{E} \mapsto \left(\left\{(P\circ(y),Q\circ(y))\in\mathrm{Eq}^{\delta}(\mathbb{T})_{X,s} \;\middle|\; (P,Q)\in\mathcal{E}_{X,Y},\ y\in Y_s\right\}\right)_{(X,s)\in\boldsymbol{\mathcal{U}}^S\times S}\end{cases}$$

$$B\begin{cases}\mathrm{Sub}(\mathrm{Eq}^{\delta}(\mathbb{T})) \longrightarrow \mathrm{Sub}(\mathrm{Eq}(\mathbb{T}))\\ \mathcal{D} \mapsto \left(\left\{(P,Q)\in\mathrm{Eq}(\mathbb{T})_{X,Y} \;\middle|\; \begin{array}{c}\forall s\in S,\ \forall (y)\colon \delta^s \longrightarrow Y,\\ (P\circ(y),Q\circ(y))\in\mathcal{D}_{X,s}\end{array}\right\}\right)_{(X,Y)\in(\boldsymbol{\mathcal{U}}^S)^2}\end{cases}$$

are order preserving. Moreover, $H\circ I = D\circ I = H\circ B = D\circ B = \mathrm{id}_{\mathrm{Sub}(\mathrm{Eq}^{\delta}(\mathbb{T}))}$ and, for every $\mathcal{E}\subseteq\mathrm{Eq}(\mathbb{T})$ and $\mathcal{D}\subseteq\mathrm{Eq}^{\delta}(\mathbb{T})$, we have that $D(\mathcal{E})\subseteq\mathcal{D}$ iff $\mathcal{E}\subseteq B(\mathcal{D})$ and $I(\mathcal{D})\subseteq\mathcal{E}$ iff $\mathcal{D}\subseteq H(\mathcal{E})$, hence $D\dashv B$ and $I\dashv H$.

From the Proposition just stated we can conclude that the unit of the adjunction $I\dashv H$ and the counit of the adjunction $D\dashv B$ are identities. Moreover, the composite adjunction $D\circ I\dashv H\circ B$ is the identity adjunction, hence the category $\underline{\mathrm{Sub}}(\mathrm{Eq}^{\delta}(\mathbb{T}))$ is a retract of $\underline{\mathrm{Sub}}(\mathrm{Eq}(\mathbb{T}))$ in the category **Adj** of categories and adjunctions.

Proposition 35. *Let $\mathbb{T}$ be a monad in $\mathbf{Set}^S$. Then the following diagrams commute*

$$\begin{array}{ccc}\underline{\mathrm{Sub}}(\mathrm{Eq}(\mathbb{T})) & \underset{\mathrm{Mod}_{\mathbb{T}}}{\overset{\mathrm{Th}_{\mathbb{T}}}{\rightleftarrows}}\ (\top) & \underline{\mathrm{Sub}}(\mathbf{EM}(\mathbb{T}))^{\mathrm{op}}\\ D\downarrow\ \dashv\ \uparrow B & & \Vert\\ \underline{\mathrm{Sub}}(\mathrm{Eq}^{\delta}(\mathbb{T})) & \underset{\mathrm{Mod}^{\delta}_{\mathbb{T}}}{\overset{\mathrm{Th}^{\delta}_{\mathbb{T}}}{\rightleftarrows}}\ (\top) & \underline{\mathrm{Sub}}(\mathbf{EM}(\mathbb{T}))^{\mathrm{op}}\end{array}\qquad\begin{array}{ccc}\underline{\mathrm{Sub}}(\mathrm{Eq}^{\delta}(\mathbb{T})) & \underset{\mathrm{Mod}^{\delta}_{\mathbb{T}}}{\overset{\mathrm{Th}^{\delta}_{\mathbb{T}}}{\rightleftarrows}}\ (\top) & \underline{\mathrm{Sub}}(\mathbf{EM}(\mathbb{T}))^{\mathrm{op}}\\ I\downarrow\ \dashv\ \uparrow H & & \Vert\\ \underline{\mathrm{Sub}}(\mathrm{Eq}(\mathbb{T})) & \underset{\mathrm{Mod}_{\mathbb{T}}}{\overset{\mathrm{Th}_{\mathbb{T}}}{\rightleftarrows}}\ (\top) & \underline{\mathrm{Sub}}(\mathbf{EM}(\mathbb{T}))^{\mathrm{op}}\end{array}$$

This fact implies that the adjunctions $\mathrm{Mod}_{\mathbb{T}}\dashv\mathrm{Th}_{\mathbb{T}}$ and $\mathrm{Mod}^{\delta}_{\mathbb{T}}\dashv\mathrm{Th}^{\delta}_{\mathbb{T}}$ are equivalent in a convenient 2-category of adjunctions, algebraic morphisms of adjunctions and deformations between algebraic morphisms of adjunctions.

For the equations in $\mathrm{Eq}^{\delta}(\mathbb{T})$ we define a closure system $\mathcal{C}^{\delta}_{\mathbb{T}}$ such that the lattices $(\mathcal{C}^{\delta}_{\mathbb{T}},\subseteq)$ and $\underline{\mathrm{Cgr}}^{\Pi}(\mathbf{Pol}(\mathbb{T}))$ are isomorphic.

Proposition 36. *Let* $\mathbb{T}$ *be a monad in* $\mathbf{Set}^S$ *and* $\mathcal{C}^{\delta}_{\mathbb{T}}$ *be the set of all those parts* $\mathcal{E}$ *of* $\mathrm{Eq}^{\delta}(\mathbb{T})$ *that satisfy the following conditions:*

(1) *For every* $(X,s) \in \boldsymbol{\mathcal{U}}^S \times S$, $\mathcal{E}_{X,s}$ *is an equivalence.*

(2) *For every* $(P,Q) \in \mathcal{E}_{Y,s}$ *and* $(P',Q') \in \mathrm{Hom}_{\mathbf{Pol}(\mathbb{T})}(X,Y)^2$, *if, for every* $t \in S$ *and* $(y)\colon \delta^t \longrightarrow Y$, $(P' \circ (y), Q' \circ (y)) \in \mathcal{E}_{X,t}$, *then* $(P \diamond P', Q \diamond Q') \in \mathcal{E}_{X,s}$.

Then $\mathcal{C}^{\delta}_{\mathbb{T}}$ *is a closure system on* $\mathrm{Eq}^{\delta}(\mathbb{T})$.

Proposition 37. *Let* $\mathbb{T}$ *be a monad in* $\mathbf{Set}^S$. *Then we have that the lattices* $(\mathcal{C}^{\delta}_{\mathbb{T}}, \subseteq)$ *and* $\underline{\mathrm{Cgr}}^{\Pi}(\mathbf{Pol}(\mathbb{T}))$ *are isomorphic.*

References

[1] J. Bénabou, *Structures algebriques dans les categories*, Cahiers de Topologie et Géometrie Différentielle, **10** (1968), pp. 1–126.

[2] G. Birkhoff, *On the structure of abstract algebras*, Proceedings of the Cambridge Philosophical Society, **31** (1935), pp. 433–454.

[3] J. Climent and J. Soliveres, *On the morphisms and transformations of Tsuyoshi Fujiwara*, manuscript, 2002.

[4] P. Cohn, *Universal Algebra*, D. Reidel Publishing Company, 1981.

[5] J. Goguen and J. Meseguer, *Completeness of many-sorted equational logic*, Houston Journal of Mathematics, **11** (1985), pp. 307–334.

[6] S. MacLane, *Categories for the working mathematician. 2nd ed.*, Springer-Verlag, 1998.

[7] R. McKenzie, G. McNulty and W. Taylor, *Algebras, lattices, varieties. Vol. I*, Wadsworth & Brooks/Cole Mathematics Series, 1987.

Universidad de Valencia, Departamento de Lógica y Filosofía de la Ciencia, E-46010 Valencia, Spain

E-mail address: Juan.B.Climent@uv.es